Thermodynamic uncertainty relations and speed limits in open quantum system

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(1) Asymptotic expansion of the solution of the master equation and its application to speed limit

GKSL equation

Quantum master equation (Gorini-Kossakowski-Sudarshan-Lindblad equation)

$$\frac{d\rho(t)}{dt} = \mathcal{L}(\alpha_t)\rho(t) = -i[H(t), \rho(t)] + \sum_k \gamma_k(t)\mathcal{D}[L_k(t)]\rho(t)$$

$$\mathcal{D}[L]\rho := L\rho L^\dagger - \frac{1}{2}\left(L^\dagger L\rho + \rho L^\dagger L\right)$$

$$\frac{d}{dt}|\rho(t)\rangle\rangle = \mathcal{L}(\alpha_t)|\rho(t)\rangle\rangle$$

$$\langle\langle A|B\rangle\rangle = \mathrm{tr}(A^\dagger B)$$

$\langle\langle 1|\mathcal{L}(\alpha) = 0$ Conservation of probability

$\mathcal{L}(\alpha)|\rho_0(\alpha)\rangle\rangle = 0$ Steady state

Dynamical steady state

$$\mathcal{R}(\alpha)\mathcal{L}(\alpha) = 1 - |\rho_0(\alpha)\rangle\rangle\langle\langle 1|$$

General inverse matrix (but it is not Moore-Penrose)

Quantum master equation $\left[1 - \mathcal{R}(\alpha_t)\frac{d}{dt}\right]|\delta\rho(t)\rangle\rangle = \mathcal{R}(\alpha_t)\frac{d}{dt}|\rho_0(\alpha_t)\rangle\rangle$ $\quad \delta\rho(t) := \rho(t) - \rho_0(\alpha_t)$

Formulaic solution $|\delta\rho^{\mathrm{dss}}(t)\rangle\rangle := \sum_{n=1}^{\infty}\left[\mathcal{R}(\alpha_t)\frac{d}{dt}\right]^n|\rho_0(\alpha_t)\rangle\rangle$

General solution $\delta\rho(t) = \delta\rho^{\mathrm{dss}}(t) + \tilde{\delta}\rho(t)$ $\quad \left[1 - \mathcal{R}(\alpha_t)\frac{d}{dt}\right]|\tilde{\delta}\rho(t)\rangle\rangle = 0$

Decreasing exponentially $\quad \tilde{\delta}\rho(0) = \delta\rho(0) - \delta\rho^{\mathrm{dss}}(0)$

Dynamical steady state $\rho^{\mathrm{dss}}(t) := \rho_0(\alpha_t) + \delta\rho^{\mathrm{dss}}(t)$

SN, M. Taguchi, T. Kubo, and Y. Tokura, PRB **92**, 195420 (2015).
D. Mandal and C. Jarzynski, J. Stat. Mech. (2016) 063204.
SN, arXiv:1710.05646 (Doctoral thesis).

Spinless one-level quantum dot

$$\frac{d}{dt}\begin{pmatrix} p_0 \\ p_1 \end{pmatrix} = \gamma \begin{pmatrix} -f & 1-f \\ f & -(1-f) \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \end{pmatrix} \qquad f = f(\beta\Delta(t)), \quad f(x) := \frac{1}{e^x + 1}$$

$$\tilde{F} := p_1^{\mathrm{dss}}(t) = \lim_{N\to\infty} \tilde{F}_{(N)}(t) \qquad \tilde{F}_{(N)}(t) := \sum_{n=0}^{N} \frac{(-1)^n}{\gamma^n} \frac{d^n}{dt^n} f(\beta\Delta(t))$$

For $\beta\Delta(t) = W + Vt$, the Borel sum of $\tilde{F}$ is

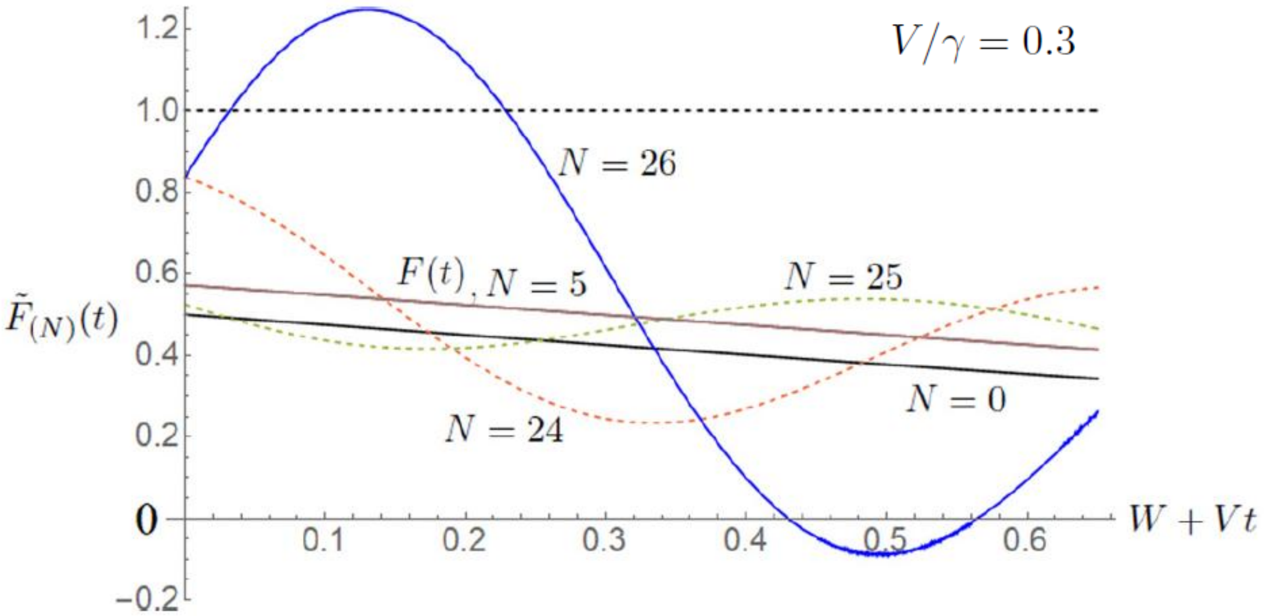
$$F(t) = f(W+Vt)\,{}_2F_1(1,1;T+1;1-f(W+Vt))$$

$$T = \frac{\gamma}{V}$$

Borel sum ↑

Asymptotic expansion of $1/T$ ↓

$$\tilde{F} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\gamma^n} \frac{d^n}{dt^n} f(W+Vt)$$



SN and Y. Utsumi, PRE **104**, 054139 (2021).

Master equation and local detailed balance

$$\frac{d}{dt}p_n(t) = \sum_m W_{nm} p_m(t)$$

$$W_{nm} = \sum_b \underline{W_{nm}^{(b)}}$$

Contribution from bath b

$$\sum_n W_{nm}^{(b)} = 0$$ Conservation of probability

Inverse temperature of bath b

Local detailed balance

$$W_{nm}^{(b)} e^{-\beta_b E_m} = W_{mn}^{(b)} e^{-\beta_b E_n} \quad (m \neq n)$$

Energy of state n

Entropy production rate

$$\frac{d}{dt}\langle E\rangle = \frac{d}{dt}\sum_n E_n p_n = \underbrace{\sum_n \frac{dE_n}{dt}p_n}_{\text{Work rate}} + \underbrace{\sum_n E_n \frac{dp_n}{dt}}_{\text{Heat current}}$$

Work rate

Heat current

$$\sum_n E_n \frac{dp_n}{dt} = \sum_n E_n \sum_b \sum_{m(\neq n)} (W_{nm}^{(b)} p_m - W_{mn}^{(b)} p_n)$$

$$= \sum_b \sum_{m\neq n} W_{nm}^{(b)} p_m (E_n - E_m)$$

Heat current form bath b to the system

$$\dot{\sigma} := \frac{d}{dt}S_{\text{Shannon}} - \sum_b \beta_b \sum_{m\neq n} W_{nm}^{(b)} p_m (E_n - E_m)$$

$$S_{\text{Shannon}} = -\sum_n p_n \ln p_n$$ Shannon entropy

Speed limit of Shiraishi-Funo-Saito

$$\sum_n \left|\frac{dp_n}{dt}\right| \le \sqrt{2\dot{\sigma}(t)a(t)}$$

$$a(t) := \sum_{n \neq m} W_{nm} p_m$$ Activity rate

$$L := \sum_n |p_n(\tau) - p_n(0)| \le \sqrt{2\sigma A}$$

$$\sigma := \int_0^\tau dt\ \dot{\sigma}(t)$$ Entropy production

$$A := \int_0^\tau dt\ a(t)$$ Activity

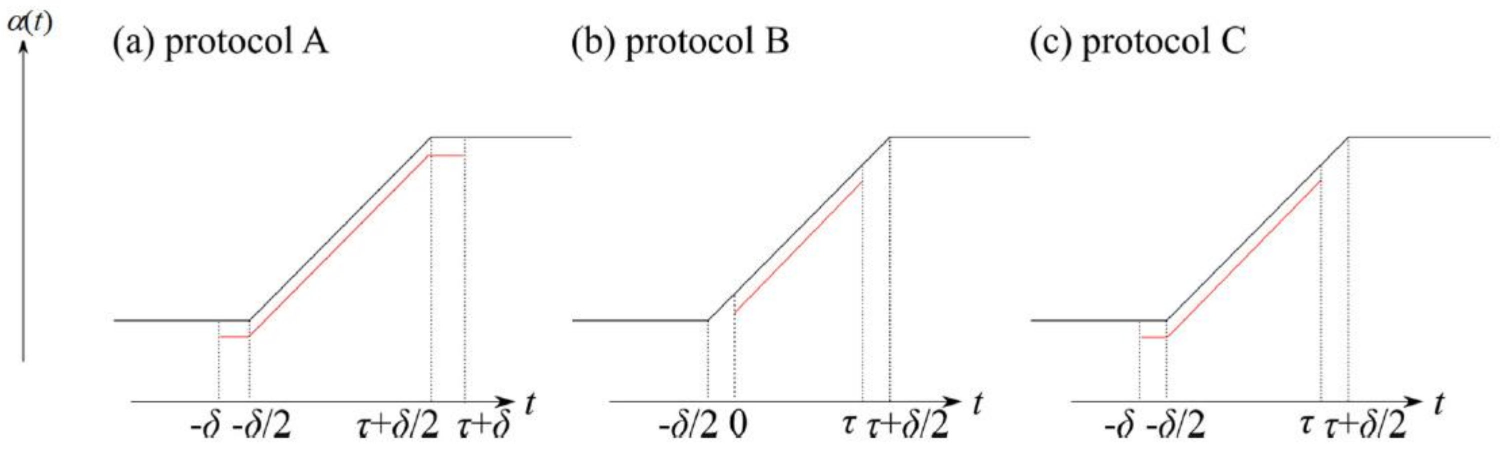
N. Shiraishi, K. Funo and K. Saito, PRL **121**, 070601 (2018).

Example of achieving the equality: a single-level quantum dot

$$\Delta(t) = \alpha(t)\Delta$$

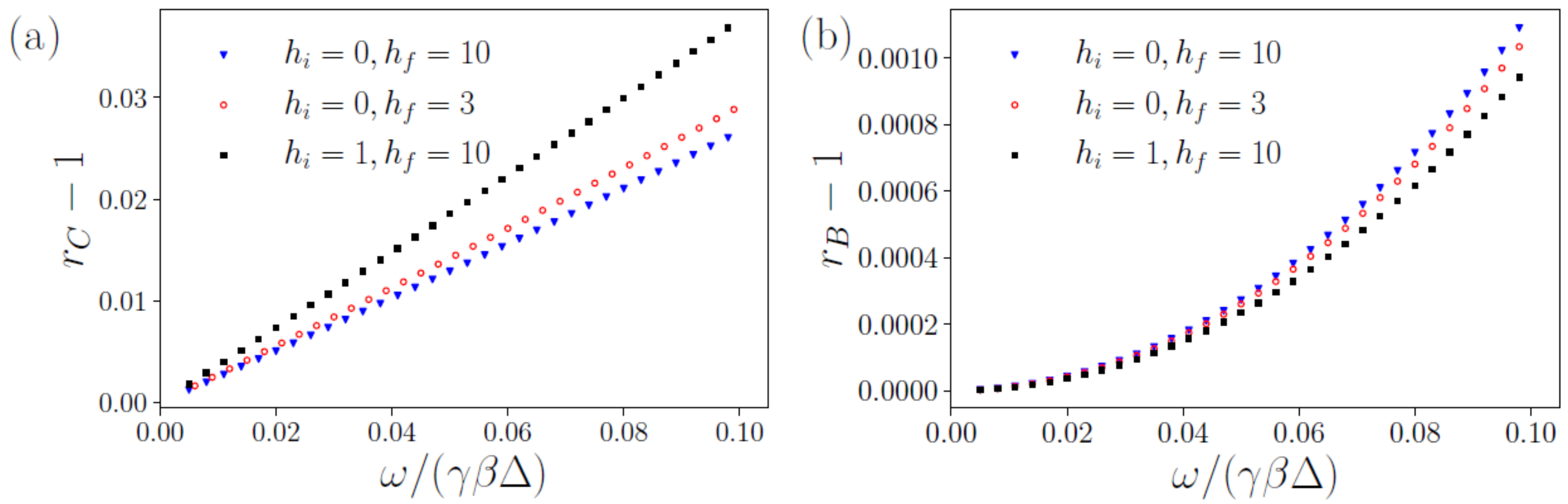
$$\alpha(t) = h_i + \omega t$$

$$h_f := h_i + \omega\tau$$



$$r := \frac{2\sigma A}{L^2}$$

$$\beta\Delta = 1$$



SN and Y. Utsumi, PRE **104**, 054139 (2021).

(2) Speed limits for open quantum system

Speed limits in closed quantum systems

$$\frac{d\rho}{dt} = -i[H(t), \rho]$$

Mandelstam-Tamm relation

$$\mathcal{L}(\rho(\tau), \rho(0)) \leq \int_0^\tau dt \; \Delta E$$

$$\mathcal{L}(\rho, \sigma) := \cos^{-1} F(\rho, \sigma),$$ Bures angle (distance)

$$F(\rho, \sigma) := \mathrm{tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}} = F(\sigma, \rho),$$ Fidelity

$$\Delta E := \sqrt{\mathrm{tr}(\rho H^2) - [\mathrm{tr}(\rho H)]^2}.$$

L. Mandelstam and I. Tamm,
"The Uncertainty Relation Between Energy and Time in Non-relativistic Quantum Mechanics"
J. Phys. (Moscow) **9**, 249 (1945).

Open quantum system

GKSL equation
$$\frac{d}{dt}\rho(t) = -i[H(t),\rho(t)] + \mathcal{D}(\rho)$$

$$\mathcal{D}(\rho) := \sum_k \gamma_k \left(\underline{L_k} \rho L_k^\dagger - \frac{1}{2} L_k^\dagger L_k \rho - \frac{1}{2} \rho L_k^\dagger L_k \right)$$

$$= \sum_b \mathcal{D}_b(\rho)$$

Jump operator

$$H(t) := H_S(t) + \underline{H_L(t)}$$

Lamb shift Hamiltonian

$$[H_L(t), H_S(t)] = 0$$

$$[L_{\underline{b},a,\underline{\omega}}, H_S] = \underline{\omega} L_{b,a,\omega}, \quad L_{b,a,-\omega} = L_{b,a,\omega}^\dagger.$$

Label of bath

Energy difference

$$\gamma_{b,a,-\omega} = e^{-\beta_b \omega} \gamma_{b,a,\omega}$$

Local detailed balance

$\gamma_k (\geq 0)$, L_k, ω and β_b can depend on time.

Entropy production rate and activity

$$\dot{\sigma} := -\operatorname{tr}\left[\frac{d\rho}{dt}\ln\rho\right] - \sum_b \beta_b \operatorname{tr}[\mathcal{D}_b(\rho) H_S]$$

Diagonalization $\rho(t) = \sum_n p_n(t)|n(t)\rangle\langle n(t)|$

Exclude terms where $n = m$ and $\omega = 0$

$$\dot{\sigma} = \frac{1}{2}\sum_{n,m,k}{}' \pi(a_{nm}^{(k)}, a_{mn}^{(-k)}) \qquad -k = (b, a, -\omega) \qquad \pi(a,b) := (a-b)\ln\frac{a}{b} \geq 0$$

$$a_{nm}^{(k)} := \gamma_k |\langle n(t)|L_k|m(t)\rangle|^2 p_m$$

$\mathfrak{a}(t) := \sum_{n\neq m}\sum_k a_{nm}^{(k)}$ Activity rate

Funo-Shiraishi-Saito

$$||\rho(\tau)-\rho(0)||_1 \le \int_0^\tau dt \left|\left|\frac{d\rho}{dt}\right|\right|_1 \qquad ||X||_1 := \mathrm{Tr}\sqrt{X^\dagger X}$$

$$\left|\left|\frac{d\rho}{dt}\right|\right|_1 \le ||[H,\rho]||_1 + ||\mathcal{D}_{\mathrm{nd}}(\rho)||_1 + ||\mathcal{D}_{\mathrm{d}}(\rho)||_1$$

$$\mathcal{D}_{\mathrm{d}}(\rho) := \sum_n |n\rangle\langle n|\mathcal{D}(\rho)|n\rangle\langle n|$$

$$\mathcal{D}_{\mathrm{nd}}(\rho) := \sum_{n\neq m} |n\rangle\langle n|\mathcal{D}(\rho)|m\rangle\langle m|$$

$$||\mathcal{D}_{\mathrm{d}}(\rho)||_1 = \sum_n \left|\frac{dp_n}{dt}\right| \le \sqrt{2\dot{\sigma}(t)\mathfrak{a}(t)}$$ Shiraishi-Funo-Saito

$$||[H,\rho]||_1 \le 2\Delta E$$ Mandelstam-Tamm-like

$$\mathcal{D}_{\mathrm{nd}}(\rho) = -i[H_D,\rho] + \sum_{n\neq m, p_n=p_m} |n\rangle\langle n|\mathcal{D}(\rho)|m\rangle\langle m|$$

$$H_D := i\sum_{p_n\neq p_m} \frac{|n\rangle\langle n|\mathcal{D}(\rho)|m\rangle\langle m|}{p_m-p_n}$$

K. Funo, N. Shiraishi and K. Saito, New. J. Phys. **21**, 013006 (2019).

Improvements to the Funo-Shiraishi-Saito

$$\left\|\frac{d\rho}{dt}\right\|_1 \leq \sqrt{\mathcal{F}_\rho(H+H_D)}\sqrt{V_\rho(\psi)} + \sqrt{2\dot{\sigma}(t)\mathfrak{m}_\rho(\psi)}$$

SLD Fisher Information

$$\psi := \mathrm{sgn}(\frac{d\rho}{dt})$$

$$\mathfrak{m}_\rho(X) := \sum_{k,n\neq m} |\langle m|X|m\rangle|^2 \Psi(a_{nm}^{(k)}, a_{mn}^{(-k)}) \qquad a_{nm}^{(k)} = \gamma_k |\langle n|L_k|m\rangle|^2 p_m$$

$$V_\rho(\psi) := \mathrm{tr}[\psi^2\rho] - (\mathrm{tr}[\psi\rho])^2 \leq 1$$

$$\mathfrak{m}_\rho(\psi) \leq \sum_{k,n\neq m} \Psi(a_{nm}^{(k)}, a_{mn}^{(-k)}) =: \mathfrak{m} \leq \mathfrak{a}$$

$$\mathcal{F}_\rho(X) \leq 4V_\rho(X)$$

$$\sqrt{\mathcal{F}_\rho(H+H_D)} \leq 2\sqrt{V_\rho(H+H_D)} \leq 2\sqrt{V_\rho(H)} + 2\sqrt{V_\rho(H_D)}$$

Logarithmic mean $\Psi(a,b) := \frac{a-b}{\ln\frac{a}{b}} \leq \frac{a+b}{2}$

$$\left|\mathrm{tr}\left[X\frac{d\rho}{dt}\right]\right| \leq \sqrt{\mathcal{F}_\rho(H+H_D)}\sqrt{V_\rho(X)} + \sqrt{2\dot{\sigma}(t)\mathfrak{m}_\rho(X)}$$

K. Sekiguchi, SN, K. Funo, H. Tajima, arXiv:2410.11604

Vu-Hasegawa and Vu-Saito

$$\sigma \geq \sigma_{\mathrm{V0}} := \frac{d_T(\rho(\tau), \rho(0))^2}{2M} \geq \sigma_{\mathrm{V1}} := \frac{d_T(\rho(\tau), \rho(0))^2}{2B}$$

$$d_T(\rho(\tau), \rho(0)) := \sum_n |q_n(\tau) - q_n(0)|$$

Semi-classical distance

$\{q_n(t)\}$ is the list of eigenvalues of ρ in ascending order

$$M := \int_0^\tau dt\ \mathfrak{m}(t)$$

$$B := \int_0^\tau dt\ b(t)$$

$$\mathfrak{m}(t) \leq \mathfrak{a}(t) \leq b(t) := \sum_k \gamma_k \mathrm{tr}[L_k^\dagger L_k \rho]$$

T. V. Vu and Y. Hasegawa, PRL **126**, 010601 (2021).

T. V. Vu and K. Saito, PRX **13**, 011013 (2023).

Nakajima-Utsumi (1)

$$\sigma \geq \sigma_0 \geq \sigma_1 \geq \sigma_2$$

$$\sigma_\alpha := \frac{||\tilde{\rho}(\tau) - \tilde{\rho}(0)||_1^2}{2A_\alpha}$$

$$\tilde{\rho}(t) := U^\dagger(t)\rho(t)U(t)$$

$$\frac{dU(t)}{dt} = -iH(t)U(t), \quad U(0) = 1$$

$$A_0 := \int_0^\tau dt\ \mathrm{tr}\Big(\tilde{\rho}(t) \sum_k \frac{1}{4}\gamma_k [\varphi, \tilde{L}_k]^\dagger [\varphi, \tilde{L}_k]\Big)$$

$$A_1 := \int_0^\tau dt\ \frac{1}{2}\Big[b(t) + \mathrm{tr}\Big(\varphi\tilde{\rho}\varphi \sum_k \gamma_k \tilde{L}_k^\dagger \tilde{L}_k\Big)\Big]$$

$$||\tilde{\rho}(\tau) - \tilde{\rho}(0)||_1 \geq d_T(\rho(\tau), \rho(0))$$

$$A_2 := \int_0^\tau dt\ \frac{1}{2}\Big[b(t) + \sum_k \gamma_k ||L_k||_\infty^2\Big]$$

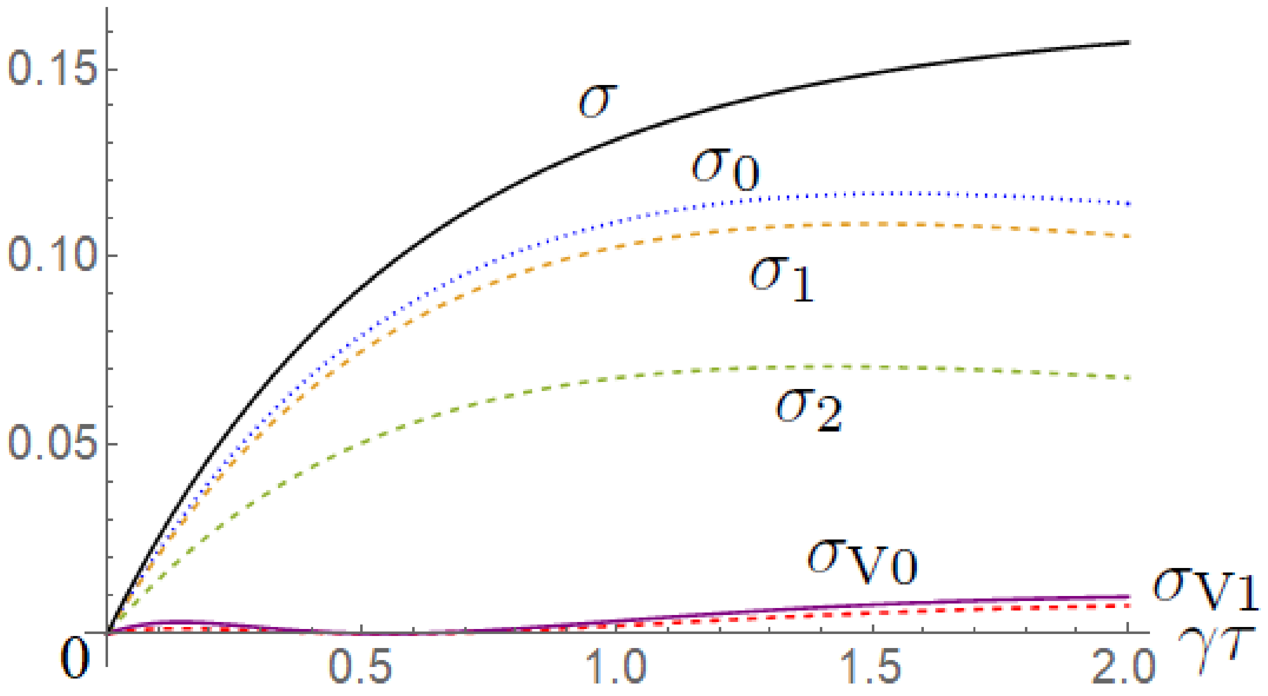
$$\varphi(t) := \mathrm{sgn}(\tilde{\rho}(t) - \tilde{\rho}(0))$$

SN and Y. Utsumi, New. J. Phys. **24**, 095004 (2022).

Comparison in quantum dot

$$\frac{d\rho}{dt} = -i[\varepsilon a^\dagger a, \rho] + \gamma[1 - f(\varepsilon)]\hat{D}[a](\rho) + \gamma f(\varepsilon)\hat{D}[a^\dagger](\rho)$$

$$f(\varepsilon) = \frac{1}{e^{\beta\varepsilon} + 1} \qquad \hat{D}[L](\rho) := \left(L\rho L^\dagger - \frac{1}{2}L^\dagger L\rho - \frac{1}{2}\rho L^\dagger L\right)$$



$\beta\varepsilon = 1$

$\rho(0) = \frac{1}{2}(1 + 0.3\tau_y)$

Pauli matrix

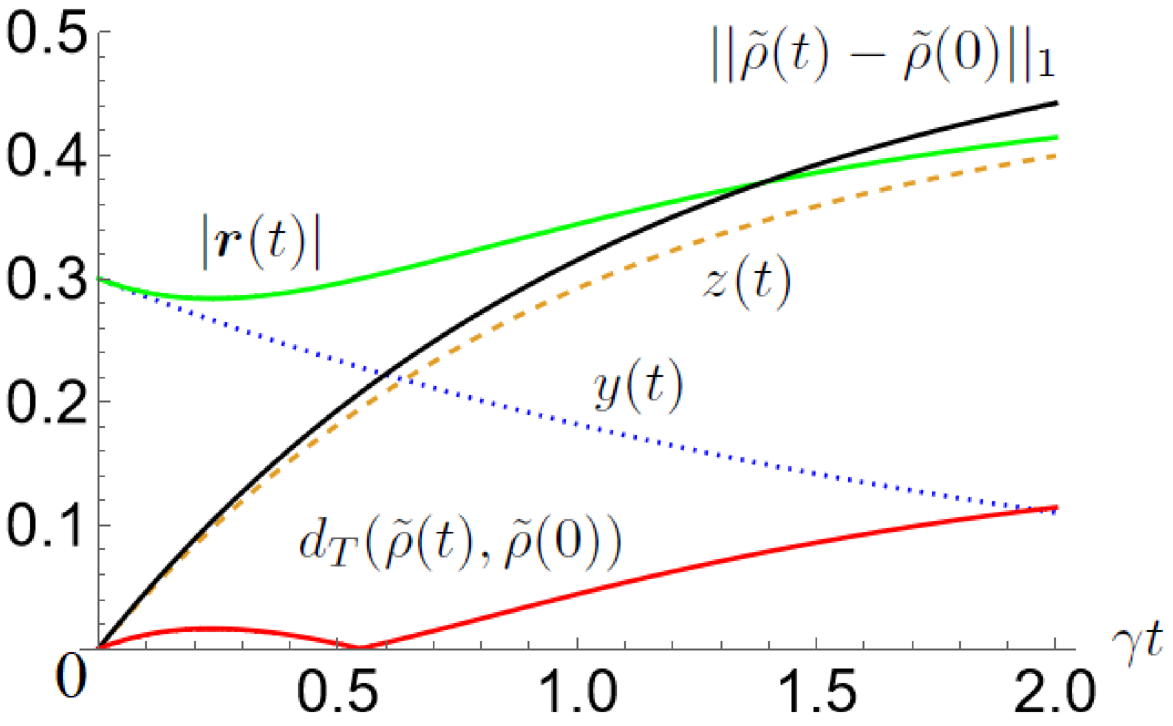
SN and Y. Utsumi, New. J. Phys. **24**, 095004 (2022).

State-to-state distances

$$\tilde{\rho}(t) = \frac{1}{2}(1 + \boldsymbol{r}(t)\cdot\boldsymbol{\tau})$$

$$||\tilde{\rho}(\tau) - \tilde{\rho}(0)||_1 = |\boldsymbol{r}(\tau) - \boldsymbol{r}(0)|, \qquad |\boldsymbol{x}| := \sqrt{\boldsymbol{x}\cdot\boldsymbol{x}}.$$

$$d_T(\rho(\tau), \rho(0)) = ||\boldsymbol{r}(\tau)| - |\boldsymbol{r}(0)|| = d_T(\tilde{\rho}(\tau), \tilde{\rho}(0)),$$



$$\beta\varepsilon = 1$$

$$\rho(0) = \frac{1}{2}(1 + 0.3\tau_y)$$

SN and Y. Utsumi, New. J. Phys. **24**, 095004 (2022).

Nakajima-Utsumi (2)

$$||\rho(\tau)-\rho(0)||_1 \le \int_0^\tau dt \left|\left|\frac{d\rho}{dt}\right|\right|_1$$
$$\le \int_0^\tau dt \left(||[H,\rho]||_1 + ||\mathcal{D}(\rho)||_1\right)$$

$$\mathcal{D}(\rho) = V^\dagger\sqrt{[\mathcal{D}(\rho)]^\dagger\mathcal{D}(\rho)} \quad (V^\dagger V = 1)$$

$$||\rho(\tau)-\rho(0)||_1 \le \int_0^\tau dt\, ||[H,\rho]||_1 + \sqrt{2\sigma A_V}$$

Shiraishi-Funo-Saito type

Mandelstam-Tamm type

$$||[H,\rho]||_1 \le \sqrt{\mathcal{F}_\rho(H)}$$

$$A_V := \int_0^\tau dt\ \mathrm{tr}\left(\rho\sum_k \frac{1}{4}\gamma_k [V,L_k]^\dagger[V,L_k]\right)$$

SN and Y. Utsumi, New. J. Phys. **24**, 095004 (2022).

Derivation of the Nakajima–Utsumi (1)

$$\{\rho\}_k(X) := \int_0^1 ds\ (\gamma_{-k}\rho)^s X(\gamma_k\rho)^{1-s} \qquad \gamma_{-k} = \gamma_{b,a,-\omega}$$

$$\langle\langle X, Y\rangle\rangle_{\tilde{\rho},k} := \mathrm{tr}\left[X^\dagger\{\tilde{\rho}\}_k(Y)\right]$$ **Semi-inner product**

$$(\langle\langle X, Y\rangle\rangle_{\tilde{\rho},k})^* = \langle\langle Y, X\rangle\rangle_{\tilde{\rho},k}$$

$$||X||^2_{\tilde{\rho},k} := \langle\langle X, X\rangle\rangle_{\tilde{\rho},k} \geq 0$$

$$\frac{d}{dt}\mathrm{tr}[f(X(t))] = \mathrm{tr}\left[f'(X(t))\frac{dX(t)}{dt}\right]$$

$$\frac{d}{dt}||\tilde{\rho}(t) - \tilde{\rho}(0)||_1 = \mathrm{tr}\left[\varphi(t)\frac{d\tilde{\rho}}{dt}\right] \qquad \varphi(t) := \mathrm{sgn}(\tilde{\rho}(t) - \tilde{\rho}(0))$$

$$= \frac{1}{2}\sum_k \langle\langle[\tilde{L}_k, \varphi(t)], [\tilde{L}_k, -\ln\tilde{\rho} - \beta_b\tilde{H}_S]\rangle\rangle_{\tilde{\rho},k}$$

Derivation of the Nakajima–Utsumi (2)

$$\frac{d}{dt}\tilde{\rho}(t) = \sum_k \gamma_k \hat{D}[\tilde{L}_k](\tilde{\rho}) = \frac{1}{2}\sum_k [\tilde{L}_k^\dagger, \{\tilde{\rho}\}_k([\tilde{L}_k, -\ln\tilde{\rho} - \beta_b \tilde{H}_S])]$$

$$||\tilde{\rho}(\tau) - \tilde{\rho}(0)||_1 = \frac{1}{2}\sum_k \int_0^\tau dt\ \langle\langle [\tilde{L}_k, \varphi(t)], [\tilde{L}_k, -\ln\tilde{\rho} - \beta_b \tilde{H}_S] \rangle\rangle_{\tilde{\rho},k}$$

$$\leq \frac{1}{2}\sum_k \int_0^\tau dt\ \sqrt{\left|\left|[\tilde{L}_k, \varphi]\right|\right|^2_{\tilde{\rho},k} \left|\left|[\tilde{L}_k, -\ln\tilde{\rho} - \beta_b \tilde{H}_S]\right|\right|^2_{\tilde{\rho},k}}$$

$$\leq \sqrt{\int_0^\tau dt\ \frac{1}{2}\sum_k \left|\left|[\tilde{L}_k, \varphi]\right|\right|^2_{\tilde{\rho},k}}$$

$\leq 2A_0$

$$\times \sqrt{\int_0^\tau dt\ \frac{1}{2}\sum_k \left|\left|[\tilde{L}_k, -\ln\tilde{\rho} - \beta_b \tilde{H}_S]\right|\right|^2_{\tilde{\rho},k}}$$

= Entropy production rate

SN and Y. Utsumi, New. J. Phys. **24**, 095004 (2022).

(3) Quantum thermodynamic uncertainty relations

Master equation and trajectory

$$\frac{d}{dt}p_n(t) = \sum_m R_{nm}(t)p_m(t) \qquad R_{nm}(t) = \sum_\nu R^\nu_{nm}(t)$$

Trajectory $\Gamma = \{n_0 \xrightarrow{\nu_1} n_1 \xrightarrow{\nu_2} n_2 \cdots \xrightarrow{\nu_N} n_N, (t_1, t_2, \cdots, t_N)\}$ $(t_0 = 0 < t_1 < t_2 < \cdots < t_N < \tau)$

$$p(\Gamma) = \exp\left(\int_{t_N}^{\tau} dt\ R_{n_N n_N}(t)\right) R^{\nu_N}_{n_N n_{N-1}}(t_N)\cdots$$
$$\times R^{\nu_2}_{n_2 n_1}(t_2)\exp\left(\int_{t_1}^{t_2} dt\ R_{n_1 n_1}(t)\right) R^{\nu_1}_{n_1 n_0}(t_1)\exp\left(\int_{0}^{t_1} dt\ R_{n_0 n_0}(t)\right) p_{n_0}(0)$$

$$p_n(\tau) = \int \mathcal{D}\Gamma\ \delta_{n_N,n}p(\Gamma)$$
$$= \sum_{N=0}^{\infty}\sum_{\nu_1,\cdots,\nu_N}\sum_{n_0,n_1,\cdots,n_N}^{n_{k+1}\neq n_k}\int_0^\tau dt_1\int_{t_1}^{\tau} dt_2\cdots\int_{t_{N-1}}^{\tau} dt_N\ \delta_{n_N,n}p(\Gamma)$$

Thermodynamic uncertainty relations

When $R_{nm}(t) = \tilde{R}(\omega t)$,

KUR $$\frac{[(\tau\partial_\tau - \omega\partial_\omega)\langle\Phi_\tau\rangle]^2}{\mathrm{Var}(\Phi_\tau)} \leq A$$ Activity

TUR $$\frac{[(\tau\partial_\tau - \omega\partial_\omega)\langle J_\tau\rangle]^2}{\mathrm{Var}(J_\tau)} \leq \frac{\sigma}{2}$$ Entropy production

$$\Phi_\tau(\Gamma) := \sum_{k=1}^{N} w^{\nu_k}_{n_k,n_{k-1}}$$

$$J_\tau(\Gamma) := \sum_{k=1}^{N} j^{\nu_k}_{n_k,n_{k-1}} \quad (j^{\nu}_{n,m} = -j^{\nu}_{m,n})$$ Current

$$\langle X\rangle := \int \mathcal{D}\Gamma\ X(\Gamma)p(\Gamma) \qquad \mathrm{Var}(X) := \langle X^2\rangle - \langle X\rangle^2$$

T. Koyuk and U. Seifert, PRL **125**, 260604 (2020).
齊藤圭司『ゆらぐ系の熱力学』サイエンス社, 2022.

Thermodynamic uncertainty relation and speed limit

TKUR

$$\frac{[(\tau\partial_\tau - \omega\partial_\omega)\langle J_\tau\rangle]^2}{\mathrm{Var}(J_\tau)} \leq \frac{\sigma^2}{4A} f\left(\frac{\sigma}{2A}\right)^{-2}$$

$f(x)$ is the Inverse function of $x\tanh x$

$$f(x) \geq \max\{x, \sqrt{x}\} \quad (x > 0)$$

$$L \leq \sigma f\left(\frac{\sigma}{2A}\right)^{-1}$$

$$L := \sum_n |p_n(\tau) - p_n(0)|$$

$$L \leq \sqrt{2\sigma A}$$ Shiraishi-Funo-Saito

$$L \leq 2A$$

V. T. Vo, T. V. Vu, and Y. Hasegawa, J. Phys. A: Math. Theor. **55**, 405004 (2022).

GKSL equation and quantum trajectory

GKSL equation $$\frac{d}{dt}\rho(t) = -i[H(t),\rho(t)] + \sum_k \mathcal{D}[L_k(t)](\rho) = \mathcal{L}(t)\rho(t)$$

$$\mathcal{D}[L](\rho) := L\rho L^\dagger - \frac{1}{2}(L^\dagger L\rho + \rho L^\dagger L)$$

$$p^\theta(\Gamma) = \mathrm{tr}[M^\theta(\Gamma)\rho(0)M^\theta(\Gamma)^\dagger]$$

$$M^\theta(\Gamma) := W^\theta(\tau,t_N)\Big(\prod_{\alpha=1}^{N} L^\theta_{k_\alpha}(t_\alpha)W^\theta(t_\alpha,t_{\alpha-1})\Big)$$

$$\frac{\partial W^\theta(t,s)}{\partial t} = \Big(-iH^\theta - \frac{1}{2}\sum_k (L_k^\theta)^\dagger L_k^\theta\Big)W^\theta(t,s)\,, \quad W^\theta(s,s) = 1$$

$$H^\theta := (1+\kappa_H\theta)H\,, \quad L_k^\theta := \sqrt{1+\kappa_k\theta}L_k$$

$$F := -\langle \partial_\theta^2 \ln p^\theta(\Gamma)\big|_{\theta=0}\rangle$$ Fisher Information

Derivation of quantum TUR and KUR

$$\frac{1}{\mathrm{Var}(\mathcal{J}_\tau)}\left[\partial_\theta\langle\mathcal{J}_\tau\rangle_\theta|_{\theta=0}\right]^2 \le F$$ Cramer–Rao inequality

$$\mathcal{J}_\tau = \Phi_\tau \quad (\mathrm{KUR})$$
$$\mathcal{J}_\tau = J_\tau \quad (\mathrm{TUR})$$

$$L_k^\theta := \sqrt{1+\kappa_k\theta}L_k \qquad H^\theta := (1+\kappa_H\theta)H$$

$$\Phi_\tau := \sum_{\alpha=1}^{N} w_{k_\alpha}$$

$$J_\tau := \sum_{\alpha=1}^{N} j_{k_\alpha}, \quad j_{-k} = -j_k$$

$$\kappa_k = 1 \quad (\mathrm{KUR})$$

$$\kappa_k = \frac{\mathrm{tr}[\rho L_k^\dagger L_k] - \mathrm{tr}[\rho L_{-k}^\dagger L_{-k}]}{\mathrm{tr}[\rho L_k^\dagger L_k] + \mathrm{tr}[\rho L_{-k}^\dagger L_{-k}]} \quad (\mathrm{TUR})$$

$$\partial_\theta\langle\mathcal{J}_\tau\rangle_\theta\big|_{\theta=0} = \langle\mathcal{J}_\tau\rangle + \mathcal{J}_* \quad (\mathcal{J}_\tau = \Phi_\tau, J_\tau),$$

$$\mathcal{J}_* := \int_0^\tau dt\,\sum_k \lambda_k \mathrm{tr}[\phi(t)L_k^\dagger L_k] \quad (\lambda_k = w_k, j_k),$$

$$\phi(t) := \frac{\partial\rho^\theta(t)}{\partial\theta}\Big|_{\theta=0}$$

Quantum TUR and KUR

Quantum KUR $\frac{[(1+\delta_\Phi)\langle\Phi_\tau\rangle]^2}{\mathrm{Var}(\Phi_\tau)} \leq B + \mathcal{Q}_{\mathrm{A}}$

Quantum TUR $\frac{[(1+\delta_J)\langle J_\tau\rangle]^2}{\mathrm{Var}(J_\tau)} \leq \frac{\sigma}{2} + \mathcal{Q}_{\mathrm{E}}$

Quantum correction term

This can only be calculated numerically

$$\delta_\Phi := \frac{\Phi_*}{\langle\Phi_\tau\rangle} \qquad B := \int_0^\tau dt \sum_k \mathrm{tr}(\rho L_k^\dagger L_k) = \langle N\rangle$$

$$\delta_J := \frac{J_*}{\langle J_\tau\rangle}$$

T. Van Vu and K. Saito, PRL **128**, 140602 (2022).

Upper Bound of Fisher Information (1)

We suppose that the number of the jump operator of the GKSL equation is M. We introduce $(M+1)$-dimensional Hilbert space $\mathcal{H}$ with an orthonormal basis $\{|m\rangle\}_{m=0}^{M}$ and a fictitious environment system E of which Hilbert space is $\mathcal{H}^{\otimes N}$. We consider a combined system of S, the ancilla system A, and E. We suppose that the initial state of the combined system is $|\tilde{\psi}(0)\rangle \otimes |0_{N-1}, \cdots, 0_1, 0_0\rangle$. Here, $|\tilde{\psi}(0)\rangle$ is the purification of $\rho(0)$ (*i.e.* $\mathrm{tr}_A[|\tilde{\psi}(0)\rangle\langle\tilde{\psi}(0)|] = \rho(0)$), and $|0_{N-1}, \cdots, 0_1, 0_0\rangle = \otimes_{i=0}^{N-1}|0\rangle_i$. For each $i = 0, 1, \cdots, N-1$, an environmental subspace i interacts with system S during the time interval $[i\Delta t, (i+1)\Delta t]$ via a unitary operator U_i.

$$
\begin{aligned}
|\psi^\theta\rangle &= U_{N-1}\cdots U_1 U_0|\tilde{\psi}(0)\rangle \otimes |0_{N-1}, \cdots, 0_1, 0_0\rangle \\
&= \sum_{m_0,\cdots,m_{N-1}=0}^{M} \Omega^\theta_{m_{N-1}} \cdots \Omega^\theta_{m_0}|\tilde{\psi}(0)\rangle \otimes |m_{N-1}, \cdots, m_1, m_0\rangle
\end{aligned}
$$

$$
\Omega_0^\theta = \left[1 + \left(-iH^\theta - \frac{1}{2}\sum_{m=1}^{M}(L_m^\theta)^\dagger L_m^\theta\right)\Delta t\right] \otimes 1_A
$$

$$
\Omega_m^\theta = L_m^\theta\sqrt{\Delta t} \otimes 1_A \quad (m = 1, \cdots, M)
$$

Upper Bound of Fisher Information (2)

The Fisher information associated with POVM (positive operator valued measure) $\mathcal{M}$ is denoted by $I(\theta,\mathcal{M})$. If we put $\mathcal{M}_0(\{m_i\}) := 1_{SA}\otimes|m_{N-1},\cdots,m_1,m_0\rangle\langle m_{N-1},\cdots,m_1,m_0|$ (1_{SA} is the identity operator of SA), the outcome is given by $\mathrm{tr}[\mathcal{M}_0(\{m_i\})|\psi^\theta\rangle\langle\psi^\theta|] = P^\theta(\{m_i\})$. Because of $F = I(0,\mathcal{M}_0)$ and the quantum Cramér-Rao theorem, we have

$$F \le \underline{\mathcal{I}} := 4\left[\langle\partial_\theta\psi^\theta|\partial_\theta\psi^\theta\rangle - \langle\partial_\theta\psi^\theta|\psi^\theta\rangle\langle\psi^\theta|\partial_\theta\psi^\theta\rangle\right]\Big|_{\theta=0}$$

SLD Fisher Information

$$\mathcal{I} \;=\; 4[\partial_{\theta_1}\partial_{\theta_2}\mathcal{C}(\theta_1,\theta_2) - \partial_{\theta_1}\mathcal{C}(\theta_1,\theta_2)\partial_{\theta_2}\mathcal{C}(\theta_1,\theta_2)]\Big|_{\theta_1=0=\theta_2}$$

$$\mathcal{C}(\theta_1,\theta_2) := \langle\psi^{\theta_2}|\psi^{\theta_1}\rangle = \mathrm{tr}_S\,\rho^{\theta_1,\theta_2}(\tau)\,,$$

$$\rho^{\theta_1,\theta_2}(\tau) := \mathrm{tr}_{AE}[|\psi^{\theta_1}\rangle\langle\psi^{\theta_2}|]$$

S. Gammelmark and K. Mølmer, PRL **112**, 170401 (2014).

Two-sided GKSL equation

$$\frac{d\rho^{\theta_1,\theta_2}(t)}{dt} = \mathcal{L}^{\theta_1,\theta_2}(t)\rho^{\theta_1,\theta_2}(t) \quad \text{under } \rho^{\theta_1,\theta_2}(0) = \rho(0)$$

$$\mathcal{L}^{\theta_1,\theta_2}(t)\bullet := -iH^{\theta_1}\bullet + \bullet\, iH^{\theta_2} + \sum_k \Big\{ L_k^{\theta_1} \bullet (L_k^{\theta_2})^\dagger - \frac{1}{2}\Big[(L_k^{\theta_1})^\dagger L_k^{\theta_1} \bullet + \bullet\, (L_k^{\theta_2})^\dagger L_k^{\theta_2} \Big] \Big\}$$

S. Gammelmark and K. Mølmer, PRL **112**, 170401 (2014).

While Vu-Saito had only evaluated the SLD Fisher information at the limit of long durations, we have provided an expression for arbitrary durations.

$$\mathcal{Q} := \mathcal{Q}_{\mathrm{A}}, \mathcal{Q}_{\mathrm{E}} \leq Q_+$$

Upper bound on the quantum correction term

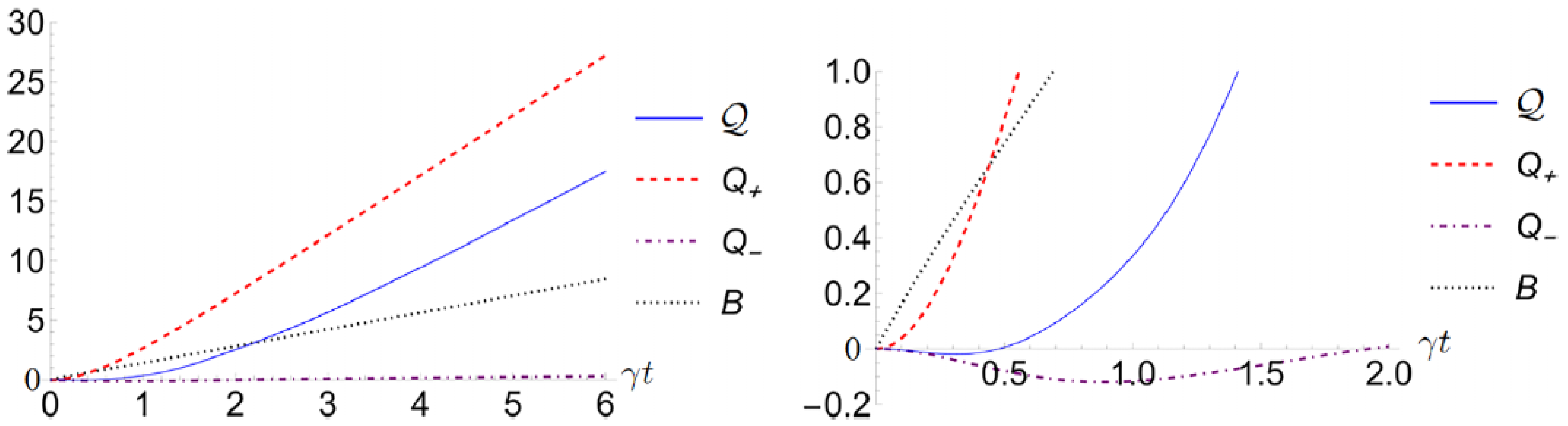
$$\mathcal{Q} \leq Q_+ = 4(I_1 + I_2 + I_3)$$

$$\begin{aligned}
I_1 &:= \int_0^\tau ds \int_0^s du \ \mathrm{tr}\Big[\mathcal{L}_2(s)\mathcal{U}(s,u)\mathcal{L}_1(u)\rho(u)\Big], \\
I_2 &:= \int_0^\tau ds \int_0^s du \ \mathrm{tr}\Big[\mathcal{L}_1(s)\mathcal{U}(s,u)\mathcal{L}_2(u)\rho(u)\Big], \\
I_3 &:= -\left(\int_0^\tau dt \ \kappa_H \mathrm{tr}\Big[H(t)\rho(t)\Big]\right)^2, \\
\mathcal{L}_1 \bullet &:= -i\kappa_H H \bullet + \frac{1}{2}\sum_k \kappa_k \Big[L_k \bullet L_k^\dagger - L_k^\dagger L_k \bullet\Big], \\
\mathcal{L}_2 \bullet &:= \bullet i\kappa_H H + \frac{1}{2}\sum_k \kappa_k \Big[L_k \bullet L_k^\dagger - \bullet L_k^\dagger L_k\Big], \\
\frac{\partial \mathcal{U}(s,u)}{\partial s} &= \mathcal{L}(s)\mathcal{U}(s,u), \quad \mathcal{U}(u,u) = 1.
\end{aligned}$$

SN and Y. Utsumi, PRE **108**, 054136 (2023).
T. Nishiyama and Y. Hasegawa, PRE **109**, 044114 (2024).

Numerical Calculation (KUR)

$$H = \Delta|1\rangle\langle 1| + \Omega(|0\rangle\langle 1| + |1\rangle\langle 0|)$$
$$L_1 = \sqrt{\gamma n}|1\rangle\langle 0|$$
$$L_2 = \sqrt{\gamma(n+1)}|0\rangle\langle 1|$$



We set $\Omega = \gamma$, $\Delta = 0.25\gamma$, $n = 1$, and $\rho(0) = \frac{1}{2}(1 + 0.2\sigma_x + 0.3\sigma_y - 0.4\sigma_z)$

SN and Y. Utsumi, PRE **108**, 054136 (2023).

Hasegawa's method (1)

$$|\Phi(t)\rangle = U(t)|\Phi(0)\rangle$$

$$|\Phi(0)\rangle := |\tilde{\psi}(0)\rangle \otimes |0\rangle, \quad \mathrm{Tr}_A[|\tilde{\psi}(0)\rangle\langle\tilde{\psi}(0)|] = \rho(0)$$

$$U(t) = \mathrm{T}\exp\left[\int_0^t ds \left\{ -iH(s)\otimes 1_B + \sum_k [L_k(s)\otimes\phi_k^\dagger(s) - L_k(s)^\dagger \otimes \phi_k(s)]\right\}\right].$$

$$[\phi_k(t), \phi_l^\dagger(s)] = \delta_{kl}\delta(t-s)$$ Quantum field

$|0\rangle$ is the vacuum state for the fields.

$$\rho(t) = \underline{\mathrm{Tr}_B}[|\Phi(t)\rangle\langle\Phi(t)|]$$

For quantum Fields and the Ansila System

Y. Hasegawa, Nat. Commun. **14**, 2828 (2023).

Hasegawa's method (2)

$$|\Psi_\tau(s;t)\rangle := V_\tau(s;t)|\Phi(0)\rangle$$

$$V_\tau(s;t) := \mathrm{T}\exp\left[\int_0^s du\left\{-i\frac{t}{\tau}H\left(\frac{t}{\tau}u\right)\otimes 1_B + \sqrt{\frac{t}{\tau}}\sum_k [L_k\left(\frac{t}{\tau}u\right)\otimes\phi_k^\dagger(u) - L_k\left(\frac{t}{\tau}u\right)^\dagger\otimes\phi_k(u)]\right\}\right]$$

$$\rho_\tau(s;t_1,t_2) := \mathrm{Tr}_B[|\Psi_\tau(s;t_1)\rangle\langle\Psi_\tau(s;t_2)|]$$

Two-sided GKSL equation $$\frac{\partial\rho_\tau(s;t_1,t_2)}{\partial s} = \mathcal{L}_\tau(s;t_1,t_2)\rho_\tau(s;t_1,t_2)$$

$$\begin{aligned}\mathcal{L}_\tau(s;t_1,t_2)\bullet &= -i\frac{t_1}{\tau}H\left(\frac{t_1}{\tau}s\right)\bullet + \bullet\, i\frac{t_2}{\tau}H\left(\frac{t_2}{\tau}s\right)\\ &+ \sqrt{\frac{t_1}{\tau}}\sqrt{\frac{t_2}{\tau}}\sum_k L_k\left(\frac{t_1}{\tau}s\right)\bullet L_k\left(\frac{t_2}{\tau}s\right)^\dagger\\ &- \frac{1}{2}\sum_k\left[\frac{t_1}{\tau}L_k\left(\frac{t_1}{\tau}s\right)^\dagger L_k\left(\frac{t_1}{\tau}s\right)\bullet + \bullet\,\frac{t_2}{\tau}L_k\left(\frac{t_2}{\tau}s\right)^\dagger L_k\left(\frac{t_2}{\tau}s\right)\right]\end{aligned}$$

Y. Hasegawa, Nat. Commun. **14**, 2828 (2023).

Hasegawa's KUR

$$\mathcal{L}_\tau(s;t,t) = \frac{t}{\tau}\mathcal{L}\left(\frac{t}{\tau}s\right) \qquad \rho_\tau(s;t,t) = \rho\left(\frac{t}{\tau}s\right)$$

KUR

$$\boxed{\frac{\tau^2\dot{\mathcal{C}}^2}{\langle\mathcal{C}^2\rangle - \langle\mathcal{C}\rangle^2} \le \mathcal{B}(\tau)}$$

$$\langle X\rangle := \langle\Psi_\tau(\tau;\tau)|X|\Psi_\tau(\tau;\tau)\rangle$$
$$= \langle\Phi(\tau)|X|\Phi(\tau)\rangle$$
$$\dot{\mathcal{C}} := \partial_t\langle\Psi_\tau(\tau;t)|\mathcal{C}|\Psi_\tau(\tau;t)\rangle\big|_{t=\tau}$$

$\mathcal{B}(t) := t^2\mathcal{J}(t)$ "The quantum generalization of the dynamical activity".

$$\mathcal{J}(t) := 4[\langle\partial_t\Psi_\tau(\tau;t)|\partial_t\Psi_\tau(\tau;t)\rangle - \langle\partial_t\Psi_\tau(\tau;t)|\Psi_\tau(\tau;t)\rangle\langle\Psi_\tau(\tau;t)|\partial_t\Psi_\tau(\tau;t)\rangle].$$

↑ Four times the energy dispersion. Quantum Fisher information.

Y. Hasegawa, Nat. Commun. **14**, 2828 (2023).

Quantum generalization of the dynamical activity

If H and L_k are time-independent,

$$\mathcal{B} = B + Q_+$$

$$Q_+ = 4(I_1 + I_2 + I_3)$$

$$\begin{aligned}
I_1 &:= \int_0^\tau ds \int_0^s du \; \mathrm{tr}\left[\mathcal{L}_2(s) e^{\mathcal{L}(s-u)} \mathcal{L}_1(u)\rho(u)\right], \\
I_2 &:= \int_0^\tau ds \int_0^s du \; \mathrm{tr}\left[\mathcal{L}_1(s) e^{\mathcal{L}(s-u)} \mathcal{L}_2(u)\rho(u)\right], \\
I_3 &:= -\left(\int_0^\tau dt \; \mathrm{tr}\left[H\rho(t)\right]\right)^2, \\
\mathcal{L}_1 \bullet &:= -iH \bullet + \frac{1}{2} \sum_k \left[L_k \bullet L_k^\dagger - L_k^\dagger L_k \bullet\right], \\
\mathcal{L}_2 \bullet &:= \bullet iH + \frac{1}{2} \sum_k \left[L_k \bullet L_k^\dagger - \bullet L_k^\dagger L_k\right].
\end{aligned}$$

SN and Y. Utsumi, PRE **108**, 054136 (2023).

Summary

(1) We investigated the asymptotic expansion of the dynamic steady state when driving a parabolic oscillator. By applying this asymptotic expansion to the speed limit, we found an example where the equality in the Shiraishi–Funo–Saito speed limit is satisfied.

SN and Y. Utsumi, PRE **104**, 054139 (2021).

(2) We investigated the speed limit of a quantum open system.

SN and Y. Utsumi, New. J. Phys. **24**, 095004 (2022).
K. Sekiguchi, SN, K. Funo, H. Tajima, arXiv:2410.11604.

(3) We investigated the thermodynamic uncertainty relation for a quantum open system and derived an analytical expression for the quantum correction term.

SN and Y. Utsumi, PRE **108**, 054136 (2023).