Poincaré algebra

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Abstract

We explain the generators of transformations in relativistic field theory.

Contents

1 Belinfante tensor

We consider a action

$$S = \int d^4x \ \mathcal{L}(\psi^A, \partial_\mu \psi^A) \tag{1.1}$$

and the Poincaré transformation

$$\delta x^\mu = \omega^\mu{}_\nu x^\nu + a^\mu, \tag{1.2}$$

$$\delta \psi^A = \frac{1}{2}\omega_{\mu\nu}(S^{\mu\nu})^A{}_B \psi^B. \tag{1.3}$$

Here, $\omega_{\mu\nu} = -\omega_{\nu\mu}$ and $(S^{\mu\nu})^A{}_B$ determines the infinitesimal transformation of the field ψ^B. a^μ and $\omega_{\mu\nu}$ are infinitesimal. Belinfante energy-momentum tensor is defined by

$$T^{\mu\nu} := \Theta^{\mu\nu} - \partial_\sigma f^{\mu\sigma\nu} \tag{1.4}$$

where

$$\Theta^\mu{}_\nu := \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^A)}\partial_\nu \psi^A - \delta^\mu_\nu \mathcal{L} \tag{1.5}$$

is the canonical energy-momentum tensor and

$$f^{\mu\sigma\nu} := \frac{1}{2}\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^A)}(S^{\sigma\nu})^A{}_B + \frac{\partial\mathcal{L}}{\partial(\partial_\sigma\psi^A)}(S^{\nu\mu})^A{}_B - \frac{\partial\mathcal{L}}{\partial(\partial_\nu\psi^A)}(S^{\mu\sigma})^A{}_B\Big)\psi^B. \tag{1.6}$$

$f^{\mu\sigma\nu} = -f^{\sigma\mu\nu}$ holds. We introduce

$$G(\sigma) := -\int_\sigma d\sigma_\mu\ T^{\mu\nu}(\omega_{\nu\lambda}x^\lambda + a_\nu). \tag{1.7}$$

Here, σ is a hyperplane. We also introduce

$$P^\nu(\sigma) := \int_\sigma d\sigma_\mu\ T^{\mu\nu}, \tag{1.8}$$

$$J^{\nu\lambda}(\sigma) := \int_\sigma d\sigma_\mu\ (T^{\mu\nu}x^\lambda - T^{\mu\lambda}x^\nu). \tag{1.9}$$

Then, we have

$$G(\sigma) = -[P^\nu(\sigma)a_\nu + \frac{1}{2}J^{\nu\lambda}(\sigma)\omega_{\nu\lambda}]. \tag{1.10}$$

We introduce

$$p_A^\mu := \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^A)}, \tag{1.11}$$

$$\pi_A := p_A^0, \tag{1.12}$$

$$P^\nu(t) := P^\nu(\sigma_t) = \int d^3x\ T^{0\nu}, \tag{1.13}$$

$$J^{\nu\lambda}(t) := J^{\nu\lambda}(\sigma_t) = \int d^3x\ (T^{0\nu}x^\lambda - T^{0\lambda}x^\nu). \tag{1.14}$$

$P^\nu(t)$ is given by

$$\begin{aligned} P^\nu(t) &= \int d^3x\ (\Theta^{0\nu} - \partial_\sigma f^{0\sigma\nu}) \\ &= \int d^3x\ (\Theta^{0\nu} - \partial_k f^{0k\nu}) \\ &= \int d^3x\ \Theta^{0\nu} =: \tilde{P}^\nu(t). \end{aligned} \tag{1.15}$$

$J^{\nu\lambda}(t)$ is given by

$$\begin{aligned} J^{\nu\lambda}(t) &:= \int d^3x\ (\Theta^{0\nu}x^\lambda - \Theta^{0\lambda}x^\nu - \partial_\sigma f^{0\sigma\nu}x^\lambda + \partial_\sigma f^{0\sigma\lambda}x^\nu) \\ &= \int d^3x\ (\Theta^{0\nu}x^\lambda - \Theta^{0\lambda}x^\nu - \partial_k f^{0k\nu}x^\lambda + \partial_k f^{0k\lambda}x^\nu) \\ &= \int d^3x\ (\Theta^{0\nu}x^\lambda - \Theta^{0\lambda}x^\nu + f^{0k\nu}\delta_k^\lambda - f^{0k\lambda}\delta_k^\nu). \end{aligned} \tag{1.16}$$

Then,

$$
\begin{aligned}
J^{ij}(t) &= \int d^3x\ (\Theta^{0i}x^j - \Theta^{0j}x^i + f^{0ki}\delta^j_k - f^{0kj}\delta^i_k) \\
&= \int d^3x\ (\Theta^{0i}x^j - \Theta^{0j}x^i + f^{0ji} - f^{0ij})
\end{aligned}
\tag{1.17}
$$

and

$$
\begin{aligned}
J^{0j}(t) &= \int d^3x\ (\Theta^{00}x^j - \Theta^{0j}x^0 + f^{0k0}\delta^j_k - f^{0kj}\delta^0_k) \\
&= \int d^3x\ (\Theta^{00}x^j - \Theta^{0j}x^0 + f^{0j0} - f^{00j}) \ \ (\because f^{00\lambda} = 0)
\end{aligned}
\tag{1.18}
$$

Thus, we have

$$
J^{\nu\lambda}(t) = \int d^3x\ (\Theta^{0\nu}x^\lambda - \Theta^{0\lambda}x^\nu + f^{0\lambda\nu} - f^{0\nu\lambda}) =: \tilde{J}^{\nu\lambda}(t). \tag{1.19}
$$

Here,

$$
\begin{aligned}
f^{\mu\nu\lambda} - f^{\mu\lambda\nu} &= \frac{1}{2}\Big(p^\mu_A(S^{\nu\lambda})^A{}_B + p^\nu_A(S^{\lambda\mu})^A{}_B - p^\lambda_A(S^{\mu\nu})^A{}_B \\
&\qquad - p^\mu_A(S^{\lambda\nu})^A{}_B - p^\lambda_A(S^{\nu\mu})^A{}_B + p^\nu_A(S^{\mu\lambda})^A{}_B\Big)\psi^B \\
&= p^\mu_A(S^{\nu\lambda})^A{}_B\psi^B.
\end{aligned}
\tag{1.20}
$$

Thus, we have

$$
\tilde{J}^{\nu\lambda}(t) = \int d^3x\ (\Theta^{0\nu}x^\lambda - \Theta^{0\lambda}x^\nu - \pi_A(S^{\nu\lambda})^A{}_B\psi^B). \tag{1.21}
$$

We put

$$
\begin{aligned}
G(t) &:= -[\tilde{P}^\nu(t)a_\nu + \frac{1}{2}\tilde{J}^{\nu\lambda}(t)\omega_{\nu\lambda}] \\
&= -\int d^3x\ \Big(\Theta^{0\nu}a_\nu + \Theta^{0\nu}x^\lambda\omega_{\nu\lambda} - \pi_A\frac{1}{2}\omega_{\nu\lambda}(S^{\nu\lambda})^A{}_B\psi^B\Big) \\
&= \int d^3x\ \Big(-\Theta^0{}_\nu\delta x^\nu + \pi_A\delta\psi^A\Big).
\end{aligned}
\tag{1.22}
$$

2 General theorem

2.1 General theory

For

$$
G(t) = \int d^3x\ \Big(-\Theta^0{}_\nu\delta x^\nu + \pi_A\delta\psi^A\Big), \tag{2.1}
$$

we prove [1, 2]

$$
\{\psi^A(x), G(t)\} = \delta_L\psi^A, \tag{2.2}
$$

$$
\{\pi_A(x), G(t)\} = -\pi_B\frac{\partial\delta\psi^B}{\partial\psi^A} - \partial_\mu\pi_A\delta x^\mu - \pi_A\partial_k(\delta x^k) + \partial_k(\delta x^0)p^k_A - \delta x^0[\mathcal{L}]_A. \tag{2.3}
$$

Here,

$$\delta_L F(x) := \delta F - \delta x^\mu \partial_\mu F, \tag{2.4}$$
$$[\mathcal{L}]_A := \frac{\partial \mathcal{L}}{\partial \psi^A} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^A)}. \tag{2.5}$$

We assume that δx^μ is independent from ψ^A and π_A, and $\delta\psi^A$ is described by only ψ^A and parameters.

First,

$$\begin{aligned}\{\psi^A(x), G(t)\} &= \Big\{\psi^A(x), \int d^3y\ \Big(- \Theta^0{}_\nu(y)\delta y^\nu + \pi_B(y)\delta\psi^B(y)\Big)\Big\}\\ &= \int d^3y\ \Big(- \{\psi^A(x), \Theta^0{}_\nu(y)\}\delta y^\nu + \{\psi^A(x), \pi_B(y)\}\delta\psi^B(y)\Big).\end{aligned} \tag{2.6}$$

Here,

$$\{\psi^A(x), \pi_B(y)\} = \delta^A_B \delta^3(\boldsymbol{x}-\boldsymbol{y}), \tag{2.7}$$
$$\begin{aligned}\{\psi^A(x), \Theta^0{}_\nu(y)\} &= \{\psi^A(x), \pi_B(y)(\partial_\nu\psi^B)(y) - \delta^0_\nu \mathcal{L}(y)\}\\ &= \delta^3(\boldsymbol{x}-\boldsymbol{y})(\partial_\nu\psi^A)(x) + \pi_B(y)\{\psi^A(x), (\partial_\nu\psi^B)(y)\} - \delta^0_\nu\{\psi^A(x), \mathcal{L}(y)\}\end{aligned} \tag{2.8}$$

and

$$\{\psi^A(x), (\partial_k\psi^B)(y)\} = 0, \tag{2.9}$$
$$\begin{aligned}\{\psi^A(x), \mathcal{L}(y)\} &= \frac{\delta\mathcal{L}(y)}{\delta\pi_A(x)}\\ &= p^k_B(y)\frac{\delta[(\partial_k\psi^B)(y)]}{\delta\pi_A(x)} + \pi_A(y)\frac{\delta[(\partial_0\psi^A)(y)]}{\delta\pi_A(x)}\\ &= \pi_B(y)\frac{\delta[(\partial_0\psi^B)(y)]}{\delta\pi_A(x)}.\end{aligned} \tag{2.10}$$

Then, we have

$$\begin{aligned}\{\psi^A(x), G(t)\} &= \delta\psi^A - \delta x^\nu\partial_\nu\psi^A + \int d^3y\ \Big(\delta y^0[\pi_B(y) - \pi_B(y)]\frac{\delta[(\partial_0\psi^B)(y)]}{\delta\pi_A(x)}\Big)\\ &= \delta\psi^A - \delta x^\nu\partial_\nu\psi^A\\ &= \delta_L\psi^A.\end{aligned} \tag{2.11}$$

Next,

$$\begin{aligned}\{\pi_A(x), G(t)\} &= \Big\{\pi_A(x), \int d^3y\ \Big(- \Theta^0{}_\nu(y)\delta y^\nu + \pi_B(y)\delta\psi^B(y)\Big)\Big\}\\ &= \int d^3y\ \Big(- \{\pi_A(x), \Theta^0{}_\nu(y)\}\delta y^\nu + \pi_B(y)\{\pi_A(x), \delta\psi^B(y)\}\Big)\\ &= -\int d^3y\ \{\pi_A(x), \pi_B(y)(\partial_\nu\psi^B)(y)\}\delta y^\nu + \int d^3y\ \delta^0_\nu\{\pi_A(x), \mathcal{L}(y)\}\delta y^\nu\\ &\quad - \pi_B(x)\frac{\partial\delta\psi^B}{\partial\psi^A}\\ &= -\int d^3y\ \pi_B(y)\{\pi_A(x), (\partial_\nu\psi^B)(y)\}\delta y^\nu + \int d^3y\ \delta^0_\nu\{\pi_A(x), \mathcal{L}(y)\}\delta y^\nu\\ &\quad - \pi_B(x)\frac{\partial\delta\psi^B}{\partial\psi^A}.\end{aligned} \tag{2.12}$$

Here,

$$
\begin{aligned}
-\int d^3y\ \pi_B(y)\{\pi_A(x), (\partial_\nu\psi^B)(y)\}\delta y^\nu &= -\int d^3y\ \pi_B(y)\{\pi_A(x), (\partial_k\psi^B)(y)\}\delta y^k \\
&\quad -\int d^3y\ \pi_B(y)\{\pi_A(x), (\partial_0\psi^B)(y)\}\delta y^0 \\
&= \int d^3y\ \pi_A(y)\delta y^k \frac{\partial}{\partial y^k}\delta^3(\boldsymbol{x}-\boldsymbol{y}) \\
&\quad -\int d^3y\ \pi_B(y)\{\pi_A(x), (\partial_0\psi^B)(y)\}\delta y^0 \\
&= -\partial_k[\pi_A(x)\delta x^k] + \int d^3y\ \pi_B(y)\frac{\delta[(\partial_0\psi^B)(y)]}{\delta\psi^A(x)}\delta y^0
\end{aligned} \tag{2.13}
$$

and

$$
\begin{aligned}
\int d^3y\ \delta^0_\nu\{\pi_A(x), \mathcal{L}(y)\}\delta y^\nu &= -\int d^3y\ \delta y^0 \frac{\delta\mathcal{L}(y)}{\delta\psi^A(x)} \\
&= -\delta x^0\frac{\partial\mathcal{L}}{\partial\psi^A} - \int d^3y\ \delta y^0 p^\mu_B(y)\frac{\delta[\partial_\mu\psi^B](y)}{\delta\psi^A(x)} \\
&= -\delta x^0\frac{\partial\mathcal{L}}{\partial\psi^A} - \int d^3y\ \delta y^0 p^k_B(y)\frac{\partial}{\partial y^k}\frac{\delta\psi^B(y)}{\delta\psi^A(x)} \\
&\quad -\int d^3y\ \delta y^0\pi_B(y)\frac{\delta[\partial_0\psi^B](y)}{\delta\psi^A(x)} \\
&= -\delta x^0\frac{\partial\mathcal{L}}{\partial\psi^A} + \partial_k[\delta x^0 p^k_A(x)] \\
&\quad -\int d^3y\ \delta y^0\pi_B(y)\frac{\delta[\partial_0\psi^B](y)}{\delta\psi^A(x)}.
\end{aligned} \tag{2.14}
$$

Thus, we have

$$
\begin{aligned}
\{\pi_A(x), G(t)\} &= -\pi_A\frac{\partial\delta\psi^B}{\partial\psi^A} - \partial_k[\pi_A\delta x^k] - \delta x^0\frac{\partial\mathcal{L}}{\partial\psi^A} + \partial_k[\delta x^0 p^k_A] \\
&= -\pi_B\frac{\partial\delta\psi^B}{\partial\psi^A} - \partial_k\pi_A\delta x^k - \pi_A\partial_k(\delta x^k) - \delta x^0\frac{\partial\mathcal{L}}{\partial\psi^A} + \partial_k(\delta x^0)p^k_A + \delta x^0\partial_k p^k_A \\
&= -\pi_B\frac{\partial\delta\psi^B}{\partial\psi^A} - \partial_\mu\pi_A\delta x^\mu + \delta x^0\partial_0\pi_A - \pi_A\partial_k(\delta x^k) - \delta x^0\frac{\partial\mathcal{L}}{\partial\psi^A} + \partial_k(\delta x^0)p^k_A + \delta x^0\partial_k p^k_A \\
&= -\pi_B\frac{\partial\delta\psi^B}{\partial\psi^A} - \partial_\mu\pi_A\delta x^\mu - \pi_A\partial_k(\delta x^k) + \partial_k(\delta x^0)p^k_A - \delta x^0[\mathcal{L}]_A.
\end{aligned} \tag{2.15}
$$

2.2 $\delta_L\pi_A$

Assuming

$$
\frac{\partial(x')}{\partial(x)}\mathcal{L}(\psi', \partial'_\mu\psi') - \mathcal{L}(\psi, \partial_\mu\psi') = \partial_\mu\chi^\mu(\psi^A), \tag{2.16}
$$

we will show that

$$\begin{aligned}\delta\pi_A &= -\pi_B\frac{\partial\delta\psi^B}{\partial\psi^A} - \pi_A\partial_k(\delta x^k) + \partial_k(\delta x^0)p_A^k \\ &= -\pi_B\frac{\partial\delta\psi^B}{\partial\psi^A} - \pi_A\partial_\mu(\delta x^\mu) + \partial_\mu(\delta x^0)p_A^\mu. \end{aligned} \tag{2.17}$$

Then, we have

$$\{\pi_A(x), G(t)\} = \delta_L\pi_A - \delta x^0[\mathcal{L}]_A. \tag{2.18}$$

We derive (2.17). From (2.16), we have

$$\mathcal{L}' := \mathcal{L}(\psi', \partial'_\mu\psi') = \mathcal{L} - \partial_\mu(\delta x^\mu)\mathcal{L} + \partial_\mu\chi^\mu. \tag{2.19}$$

Then,

$$\begin{aligned}\delta p_A^\nu &= \frac{\partial\mathcal{L}'}{\partial(\partial'_\nu\psi'^A)} - \frac{\partial\mathcal{L}}{\partial(\partial_\nu\psi^A)} \\ &= p_B^\mu\frac{\partial(\partial_\mu\psi^B)}{\partial(\partial'_\nu\psi'^A)}[1-\partial_\lambda(\delta x^\lambda)] - p_A^\nu. \end{aligned} \tag{2.20}$$

Because of

$$\delta_L\partial_\mu\psi^A\psi^A = \partial_\mu\delta_L\psi^A, \tag{2.21}$$

we have

$$\begin{aligned}\delta\partial_\mu\psi^A &= \delta_L\partial_\mu\psi^A + \delta x^\nu\partial_\nu\partial_\mu\psi^A \\ &= \partial_\mu(\delta\psi^A - \delta x^\nu\partial_\nu\psi^A) + \delta x^\nu\partial_\nu\partial_\mu\psi^A \\ &= \partial_\mu\delta\psi^A - \partial_\mu(\delta x^\nu)\partial_\nu\psi^A \\ &= \partial_\mu\psi^B\frac{\partial\delta\psi^A}{\partial\psi^B} - \partial_\mu(\delta x^\nu)\partial_\nu\psi^A \end{aligned} \tag{2.22}$$

and

$$\begin{aligned}\partial_\mu\psi^B &= \partial'_\mu\psi'^B - \delta\partial_\mu\psi^B \\ &= \partial'_\mu\psi'^B - \partial_\mu\psi^C\frac{\partial\delta\psi^B}{\partial\psi^C} + \partial_\mu(\delta x^\nu)\partial_\nu\psi^B \\ &= \partial'_\mu\psi'^B - \partial'_\mu\psi'^C\frac{\partial\delta\psi^B}{\partial\psi^C} + \partial_\mu(\delta x^\nu)\partial'_\nu\psi'^B + O(\delta^2). \end{aligned} \tag{2.23}$$

Thus, we have

$$\frac{\partial(\partial_\mu\psi^B)}{\partial(\partial'_\nu\psi'^A)} = \delta_A^B\delta_\mu^\nu - \delta_\mu^\nu\frac{\partial\delta\psi^B}{\partial\psi^A} + \delta_A^B\partial_\mu(\delta x^\lambda)\delta_\lambda^\nu. \tag{2.24}$$

Then, we have

$$\begin{aligned}\delta p_A^\nu &= p_B^\mu[\delta_A^B\delta_\mu^\nu - \delta_\mu^\nu\frac{\partial\delta\psi^B}{\partial\psi^A} + \delta_A^B\partial_\mu(\delta x^\nu)][1-\partial_\lambda(\delta x^\lambda)] - p_A^\nu \\ &= -p_B^\nu\frac{\partial\delta\psi^B}{\partial\psi^A} + p_A^\mu\partial_\mu(\delta x^\nu) - p_A^\nu\partial_\mu(\delta x^\mu). \end{aligned} \tag{2.25}$$

For $\nu = 0$, it becomes (2.17).

3 Poincaré algebra

3.1 General theory

(2.3) was

$$\{\pi_A(x), G(t)\} = -\pi_B \frac{\partial \delta\psi^B}{\partial \psi^A} - \partial_\mu \pi_A \delta x^\mu - \pi_A \partial_k(\delta x^k) + \partial_k(\delta x^0) p_A^k - \delta x^0 [\mathcal{L}]_A. \tag{3.1}$$

For the Poincaré transformation, it becomes

$$\{\pi_A(x), G(t)\} = -\frac{1}{2}\omega_{\alpha\beta}\pi_B (S^{\alpha\beta})^B{}_A - \partial_\mu \pi_A (a^\mu + \omega^\mu{}_\nu x^\nu) + \omega^0{}_k p_A^k - (a^0 + \omega^0{}_k x^k)[\mathcal{L}]_A. \tag{3.2}$$

From the above equation and

$$\{\psi^A(x), G(t)\} = -\frac{1}{2}\omega_{\mu\nu}(S^{\mu\nu})^A{}_B \psi^B - (a^\mu + \omega^\mu{}_\nu x^\nu)\partial_\mu \psi^A, \tag{3.3}$$

we have

$$\{\psi^A, P_\mu\} = \partial_\mu \psi^A, \tag{3.4}$$
$$\{\pi_A, P_\mu\} = \partial_\mu \pi_A + \delta^0_\mu [\mathcal{L}]_A, \tag{3.5}$$
$$\{\psi^A, J^{\mu\nu}\} = (S^{\mu\nu})^A{}_B \psi^B + (x^\nu \partial^\mu - x^\mu \partial^\nu)\psi^A, \tag{3.6}$$
$$\{\pi_A, J^{\mu\nu}\} = (S^{\mu\nu})^B{}_A \pi_B + (x^\nu \partial^\mu - x^\mu \partial^\nu)\pi_A + \Delta_A^{\mu\nu}. \tag{3.7}$$

Here,

$$\Delta_A^{ij} = 0, \tag{3.8}$$
$$\Delta_A^{0k} = p_A^k - x^k [\mathcal{L}]_A. \tag{3.9}$$

We have

$$\{P_i, P_j\} = 0, \tag{3.10}$$
$$\{J^{ij}, P^k\} = \eta^{ik} P^j - \eta^{jk} P^i, \tag{3.11}$$
$$\{J^{ij}, J^{kl}\} = \eta^{ik} J^{jl} - \eta^{il} J^{jk} - \eta^{jk} J^{il} + \eta^{jl} J^{ik} \tag{3.12}$$

and

$$\{P_j, J^{0k}\} = \delta_j^k P_0. \tag{3.13}$$

3.2 On shell

Using the EMO($[\mathcal{L}]_A = 0$) and

$$\partial_\mu \Theta^{\mu\nu} = 0, \tag{3.14}$$

we have

$$\{P_0, P_i\} = 0. \tag{3.15}$$

Using the EMO and $\partial_\mu T^{\mu\nu} = 0$ and

$$\partial_\lambda M^{\lambda\mu\nu} = T^{\mu\nu} - T^{\nu\mu} = 0, \tag{3.16}$$
$$M^{\lambda\mu\nu} := x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu}, \tag{3.17}$$

we have [2]

$$\{P_0, J^{ij}\} = 0 \tag{3.18}$$

and

$$\{P_0, J^{0k}\} = P^k. \tag{3.19}$$

In the general theory, we can not calculate $\{J^{0i}, J^{0k}\}$ and $\{J^{ij}, J^{0k}\}$ [2].

References

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[2] Weinberg S, The Quantum Theory of Fields (Volume 1), Cambridge University Press, 2005.