Gravitational energy pseudotensors

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Abstract

This note explains the gravitational energy pseudotensor and superpotentials. I mainly referred to Refs.[1, 2, 3].

Contents

1 General Theory

1.1 Notation and the Einstein Equation

In this article we consider a D-dimensional spacetime, and take the signature of the metric $g_{\mu\nu}$ to be $(-++\cdots+)$. We also define

$$\Gamma^{\sigma}{}_{\mu\nu} := \frac{1}{2}g^{\sigma\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}), \tag{1.1}$$

$$R^{\mu}{}_{\lambda\alpha\beta} := \partial_\alpha \Gamma^{\mu}{}_{\lambda\beta} - \partial_\beta \Gamma^{\mu}{}_{\lambda\alpha} + \Gamma^{\mu}{}_{\rho\alpha}\Gamma^{\rho}{}_{\lambda\beta} - \Gamma^{\mu}{}_{\rho\beta}\Gamma^{\rho}{}_{\lambda\alpha}, \tag{1.2}$$

$$R_{\mu\nu} := R^{\lambda}{}_{\mu\lambda\nu}, \tag{1.3}$$

$$R := g^{\mu\nu}R_{\mu\nu}. \tag{1.4}$$

Here,

$$\sqrt{-g}R = \sqrt{-g}G + \partial_\mu \boldsymbol{D}^\mu, \tag{1.5}$$

$$G := g^{\mu\nu}\Big[\Gamma^\rho{}_{\gamma\nu}\Gamma^\gamma{}_{\mu\rho} - \Gamma^\rho{}_{\gamma\rho}\Gamma^\gamma{}_{\mu\nu}\Big], \tag{1.6}$$

$$\boldsymbol{D}^\rho := \sqrt{-g}\Big[g^{\mu\nu}\Gamma^\rho{}_{\mu\nu} - g^{\mu\rho}\Gamma^\nu{}_{\mu\nu}\Big], \tag{1.7}$$

and $g := \det(g_{\mu\nu})$. In what follows,

$$\boldsymbol{\mathcal{L}}_\mathrm{G} := \frac{1}{2\kappa}\sqrt{-g}R, \tag{1.8}$$

$$\tilde{\boldsymbol{\mathcal{L}}}_\mathrm{G} := \frac{1}{2\kappa}\boldsymbol{G}, \quad \boldsymbol{G} := \sqrt{-g}G \tag{1.9}$$

where κ is the Einstein constant. Now, we put

$$G_{\mu\nu} := \frac{1}{\sqrt{-g}}\Big[\frac{\partial \boldsymbol{G}}{\partial g^{\mu\nu}} - \partial_\lambda \frac{\partial \boldsymbol{G}}{\partial(\partial_\lambda g^{\mu\nu})}\Big]. \tag{1.10}$$

Then,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R. \tag{1.11}$$

Also, we put

$$\boldsymbol{G}^{\mu\nu} := \frac{\partial \boldsymbol{G}}{\partial g_{\mu\nu}} - \partial_\lambda \frac{\partial \boldsymbol{G}}{\partial(\partial_\lambda g_{\mu\nu})}. \tag{1.12}$$

Then,

$$\boldsymbol{G}^{\mu\nu} = -\sqrt{-g}G^{\mu\nu}. \tag{1.13}$$

The action of the combined system of the gravitational field and matter fields (including gauge fields) is

$$S = \int d^D x \; (\boldsymbol{\mathcal{L}}_\mathrm{G} + \sqrt{-g}\mathcal{L}_\mathrm{mat}) \tag{1.14}$$

Here, $\mathcal{L}_\mathrm{mat}$ is the Lagrangian density of the matter fields. The Einstein equation is

$$\boldsymbol{G}^{\mu\nu} = \kappa \boldsymbol{T}^{\mu\nu}, \tag{1.15}$$

$$\boldsymbol{T}^{\mu\nu} := -2\Big[\frac{\partial(\sqrt{-g}\mathcal{L}_\mathrm{mat})}{\partial g_{\mu\nu}} - \partial_\lambda \frac{\partial(\sqrt{-g}\mathcal{L}_\mathrm{mat})}{\partial(\partial_\lambda g_{\mu\nu})}\Big]. \tag{1.16}$$

Here, $T^{\mu\nu} := \boldsymbol{T}^{\mu\nu}/\sqrt{-g}$ is the energy-momentum tensor.

1.2 Gravitational Energy Pseudotensor

As shown in §1.3,

$$\partial_\mu \boldsymbol{G}^\mu{}_\nu - \frac{1}{2}\partial_\nu g_{\alpha\beta}\boldsymbol{G}^{\alpha\beta} \equiv 0 \tag{1.17}$$

holds. Here, $\equiv$ denotes an equation that holds without using the equations of motion. Similarly,

$$\partial_\mu \boldsymbol{T}^\mu{}_\nu - \frac{1}{2}\partial_\nu g_{\alpha\beta}\boldsymbol{T}^{\alpha\beta} = 0 \tag{1.18}$$

is obtained. This expresses the conservation law for energy and momentum.

Clearly,

$$\partial_\mu \boldsymbol{T}^\mu{}_\nu \neq 0. \tag{1.19}$$

If there exists a quantity $\boldsymbol{t}^\mu{}_\nu$ or $\boldsymbol{t}^{\mu\nu}$ such that

$$\partial_\mu[(\sqrt{-g})^\alpha(\boldsymbol{T}^\mu{}_\nu + \boldsymbol{t}^\mu{}_\nu)] = 0 \quad (\alpha = -1, 0, 1) \tag{1.20}$$

or

$$\partial_\mu[(\sqrt{-g})^\alpha(\boldsymbol{T}^{\mu\nu} + \boldsymbol{t}^{\mu\nu})] = 0 \quad (\alpha = -1, 0, 1), \tag{1.21}$$

then

$$P_\nu := \int_{V_t} d^{D-1}x\ (\sqrt{-g})^\alpha(\boldsymbol{T}^0{}_\nu + \boldsymbol{t}^0{}_\nu) \tag{1.22}$$

or

$$P^\nu := \int_{V_t} d^{D-1}x\ (\sqrt{-g})^\alpha(\boldsymbol{T}^{0\nu} + \boldsymbol{t}^{0\nu}) \tag{1.23}$$

is a conserved quantity, as will be shown below. Here, we have chosen a coordinate system in which $t = x^0$ can be interpreted as time. V_t is the hypersurface on which the time is t. $d^{D-1}x = dx^1dx^2\cdots dx^{D-1}$. We call $\boldsymbol{t}^\mu{}_\nu$ and $\boldsymbol{t}^{\mu\nu}$ the gravitational energy pseudotensors [1)]. In the "telephone directory" (Wheeler, Misner, and Thorne), an energy pseudotensor satisfying (1.21) for $\alpha = -1$ is discussed. The Landau–Lifshitz energy pseudotensor in *The Classical Theory of Fields* satisfies (1.21) for $\alpha = 1$. In this article we treat the case in which (1.20) holds for $\alpha = 0$.

Suppose that a quantity $\boldsymbol{J}^\mu{}_r$ (where r is a label, which may or may not be a tensor index) satisfies

$$\partial_\mu \boldsymbol{J}^\mu{}_r = 0 \tag{1.24}$$

and assume the following:

[1)] An energy pseudotensor does not transform as a tensor under a general coordinate transformation, but it does transform as a tensor under the affine transformation (1.43).

- At spatial infinity, the spacetime approaches Minkowski spacetime asymptotically.
- $\boldsymbol{J}^{\mu}_{\ r}$ converges to zero sufficiently rapidly at infinity.

Then,

$$Q_r := \int d^{D-1}x\ \boldsymbol{J}^{0}_{\ r} \tag{1.25}$$

is constant. Here, the left-hand side is the integral

$$Q_r(\sigma) := \int_{\sigma} d\sigma_{\mu}\ \boldsymbol{J}^{\mu}_{\ r} \tag{1.26}$$

with the surface σ chosen to be the surface V_t on which the time $t(= x^0)$. Here, $d\sigma_{\mu}$ is the surface element, and $d\sigma_0 = d^{D-1}x = dx^1 dx^2 \cdots dx^{D-1}$. Let σ_{12} be the surface that connects the constant-time surfaces V_{t_1} and V_{t_2} (the side surface at infinity). Let Ω be the region bounded by V_{t_1}, V_{t_2}, and σ_{12}, and set $\partial\Omega = V_{t_1} + \sigma_{12} - V_{t_2}$. Then

$$Q_r(\partial\Omega) = Q_r(V_{t_1}) - Q_r(V_{t_2}) + Q_r(\sigma_{12}). \tag{1.27}$$

On the other hand, by Stokes' theorem (Gauss' theorem),

$$Q_r(\partial\Omega) = \int_{\Omega} d^D x\ \partial_{\mu}\boldsymbol{J}^{\mu}_{\ r} = 0. \tag{1.28}$$

Therefore, assuming $Q_r(\sigma_{12}) = 0$,

$$Q_r(V_t) = \int_{V_t} d\sigma_{\mu}\ \boldsymbol{J}^{\mu}_{\ r} = \int_{V_t} d^{D-1}x\ \boldsymbol{J}^{0}_{\ r} \tag{1.29}$$

is independent of time; that is, (1.25) is independent of time.

Suppose there exists a quantity that satisfies

$$(\sqrt{-g})^{\alpha}(\frac{1}{\kappa}\boldsymbol{G}^{\mu}_{\ \nu} + \boldsymbol{t}^{\mu}_{\ \nu}) \equiv \partial_{\lambda}\boldsymbol{U}^{\lambda\mu}_{\ \ \ \nu} \tag{1.30}$$

and

$$\partial_{\mu}\partial_{\lambda}\boldsymbol{U}^{\lambda\mu}_{\ \ \ \nu} = 0. \tag{1.31}$$

Then, by the Einstein equation,

$$(\sqrt{-g})^{\alpha}(\boldsymbol{T}^{\mu}_{\ \nu} + \boldsymbol{t}^{\mu}_{\ \nu}) = \partial_{\lambda}\boldsymbol{U}^{\lambda\mu}_{\ \ \ \nu} \tag{1.32}$$

and hence

$$\partial_{\mu}[(\sqrt{-g})^{\alpha}(\boldsymbol{T}^{\mu}_{\ \nu} + \boldsymbol{t}^{\mu}_{\ \nu})] = 0 \tag{1.33}$$

are obtained. If

$$\boldsymbol{U}^{\lambda\mu}{}_{\nu} \equiv -\boldsymbol{U}^{\mu\lambda}{}_{\nu}, \tag{1.34}$$

then (1.31) also holds. A quantity satisfying (1.30) and (1.34) is called a superpotential. Here, a quantity satisfying (1.30) and (1.31) will provisionally be called a pseudo-superpotential. The same discussion applies to $\boldsymbol{t}^{\mu\nu}$ (in that case, the indices of $\boldsymbol{U}$ become $\boldsymbol{U}^{\lambda\mu\nu}$).

When a superpotential exists, P_ν is

$$\begin{aligned} P_\nu &= \int_{V_t} d^{D-1}x\ \partial_\lambda \boldsymbol{U}^{\lambda 0}{}_{\nu} \\ &= \int_{V_t} d^{D-1}x\ \partial_k \boldsymbol{U}^{k 0}{}_{\nu} \\ &= \int_{S_t} dS_k\ \boldsymbol{U}^{k 0}{}_{\nu} \end{aligned} \tag{1.35}$$

and can therefore be written as a surface integral.

In §1.3, for the Einstein energy pseudotensor

$$\boldsymbol{t}^{\mu}{}_{\nu} := \frac{1}{2\kappa}\Big(\frac{\partial \boldsymbol{G}}{\partial(\partial_\mu g_{\alpha\beta})}\partial_\nu g_{\alpha\beta} - \delta^\mu_\nu \boldsymbol{G}\Big), \tag{1.36}$$

we give

$$\frac{1}{\kappa}\boldsymbol{G}^{\mu}{}_{\nu} + \boldsymbol{t}^{\mu}{}_{\nu} \equiv \partial_\lambda \boldsymbol{U}^{\lambda\mu}{}_{\nu} \equiv \partial_\lambda {}^{(0)}\boldsymbol{C}^{\lambda\mu}{}_{\nu}, \tag{1.37}$$

$$\boldsymbol{U}^{\lambda\mu}{}_{\nu} \equiv -\boldsymbol{U}^{\lambda\mu}{}_{\nu}, \tag{1.38}$$

$$\partial_\mu \partial_\lambda {}^{(0)}\boldsymbol{C}^{\lambda\mu}{}_{\nu} \equiv 0, \tag{1.39}$$

$${}^{(0)}\boldsymbol{C}^{\lambda\mu}{}_{\nu} \not\equiv -{}^{(0)}\boldsymbol{C}^{\mu\lambda}{}_{\nu} \tag{1.40}$$

with a superpotential $\boldsymbol{U}^{\lambda\mu}{}_{\nu}$ and a pseudo-superpotential ${}^{(0)}\boldsymbol{C}^{\mu\lambda}{}_{\nu}$.

1.3 Invariant Variational Theory: Derivation of the Superpotential

In this section, we introduce the superpotential by applying Noether's second theorem.

Noether's theorem consists of a first theorem for global transformations and a second theorem for local transformations. The first theorem is used frequently in textbooks on field theory. Major applications of the second theorem are gauge theory and the energy of the gravitational field explained here. The paper [4] gives a detailed discussion of these two applications.

Many studies of the gravitational energy pseudotensor seem to involve either enormous computations or flashes of genius. In this article, we select and explain those parts for which neither is needed (at least not very much). I think that the method based on Noether's second theorem is such a method.

Now, we put

$$S_{\mathrm{G}} := \int d^D x\ \boldsymbol{\mathcal{L}}_{\mathrm{G}}, \tag{1.41}$$

$$\tilde{S}_{\mathrm{G}} := \int d^D x\ \tilde{\boldsymbol{\mathcal{L}}}_{\mathrm{G}}. \tag{1.42}$$

The action S_{G} is invariant under general coordinate transformations, while $\tilde{S}_{\mathrm{G}}$ is invariant under the affine transformation

$$x^\mu \to x'^\mu = a^\mu{}_\nu x^\nu + b^\mu. \tag{1.43}$$

Here, $a^\mu{}_\nu$ and b^μ are constants, and the matrix $a^\mu{}_\nu$ is assumed to be invertible. Now consider the infinitesimal general coordinate transformation

$$x'^\mu = x^\mu + \xi^\mu(x). \tag{1.44}$$

Then the variation of S_{G} is

$$\begin{aligned} \delta S_{\mathrm{G}} &= \int d^D x \, \left[\frac{\partial(x')}{\partial(x)}\boldsymbol{\mathcal{L}}'_{\mathrm{G}}(x') - \boldsymbol{\mathcal{L}}_{\mathrm{G}}(x)\right] \\ &= \int d^D x \, (\delta\boldsymbol{\mathcal{L}}_{\mathrm{G}} + \boldsymbol{\mathcal{L}}_{\mathrm{G}}\partial_\mu\xi^\mu). \end{aligned} \tag{1.45}$$

Here,

$$\delta F(x) := F'(x') - F(x). \tag{1.46}$$

Now, we put

$$\bar{\delta} F(x) := F'(x')\big|_{x'=x} - F(x). \tag{1.47}$$

Then,

$$\begin{aligned} \delta F(x) &= F'(x') - F'(x')\big|_{x'=x} + F'(x')\big|_{x'=x} - F(x) \\ &= \partial_\mu F \delta x^\mu + \bar{\delta} F(x) \end{aligned} \tag{1.48}$$

and

$$\bar{\delta}(\partial_\mu F) = \partial_\mu(\bar{\delta} F), \tag{1.49}$$

$$\begin{aligned} \delta(\partial_\mu F) &= \bar{\delta}(\partial_\mu F) + \partial_\nu\partial_\mu F \delta x^\nu \\ &= \partial_\mu(\bar{\delta} F) + \partial_\nu\partial_\mu F \delta x^\nu \\ &= \partial_\mu(\delta F) - \partial_\nu F \partial_\mu(\delta x^\nu) \end{aligned} \tag{1.50}$$

follow. In the third equality, we used (1.48). Therefore,

$$\delta\boldsymbol{\mathcal{L}}_{\mathrm{G}} + \boldsymbol{\mathcal{L}}_{\mathrm{G}}\partial_\mu\xi^\mu = \bar{\delta}\boldsymbol{\mathcal{L}}_{\mathrm{G}} + \partial_\mu(\boldsymbol{\mathcal{L}}_{\mathrm{G}}\xi^\mu). \tag{1.51}$$

Thus,

$$\delta S_{\mathrm{G}} = \int d^D x \, \left[\bar{\delta}\boldsymbol{\mathcal{L}}_{\mathrm{G}} + \partial_\mu(\boldsymbol{\mathcal{L}}_{\mathrm{G}}\xi^\mu)\right]. \tag{1.52}$$

Here,

$$
\begin{aligned}
2\kappa\bar{\delta}\boldsymbol{\mathcal{L}}_{\rm G} &= \bar{\delta}\boldsymbol{G} + \partial_\mu \bar{\delta}\boldsymbol{D}^\mu \\
&= \frac{\partial \boldsymbol{G}}{\partial g_{\alpha\beta}}\bar{\delta}g_{\alpha\beta} + \frac{\partial \boldsymbol{G}}{\partial(\partial_\gamma g_{\alpha\beta})}\partial_\gamma \bar{\delta}g_{\alpha\beta} \\
&\quad + \partial_\mu\Big[\frac{\partial \boldsymbol{D}^\mu}{\partial g_{\alpha\beta}}\bar{\delta}g_{\alpha\beta} + \frac{\partial \boldsymbol{D}^\mu}{\partial(\partial_\gamma g_{\alpha\beta})}\partial_\gamma \bar{\delta}g_{\alpha\beta}\Big] \\
&= \boldsymbol{G}^{\alpha\beta}\bar{\delta}g_{\alpha\beta} + \partial_\mu\Big[\frac{\partial \boldsymbol{G}}{\partial(\partial_\mu g_{\alpha\beta})}\bar{\delta}g_{\alpha\beta} + \frac{\partial \boldsymbol{D}^\mu}{\partial g_{\alpha\beta}}\bar{\delta}g_{\alpha\beta} + \frac{\partial \boldsymbol{D}^\mu}{\partial(\partial_\gamma g_{\alpha\beta})}\partial_\gamma \bar{\delta}g_{\alpha\beta}\Big]. \qquad (1.53)
\end{aligned}
$$

Also,

$$
\bar{\delta}g_{\alpha\beta} = -\partial_\alpha \xi^\lambda g_{\lambda\beta} - \partial_\beta \xi^\lambda g_{\lambda\alpha} - \xi^\mu \partial_\mu g_{\alpha\beta}, \qquad (1.54)
$$

therefore

$$
\begin{aligned}
\boldsymbol{G}^{\alpha\beta}\bar{\delta}g_{\alpha\beta} &= -2\partial_\alpha \xi^\lambda \boldsymbol{G}^\alpha{}_\lambda - \xi^\mu \partial_\mu g_{\alpha\beta}\boldsymbol{G}^{\alpha\beta} \\
&= \partial_\mu[-2\xi^\lambda \boldsymbol{G}^\mu{}_\lambda] + 2\xi^\mu \partial_\alpha \boldsymbol{G}^\alpha{}_\mu - \xi^\mu \boldsymbol{G}^\alpha{}_\mu \partial_\mu g_{\alpha\beta}\boldsymbol{G}^{\alpha\beta}. \qquad (1.55)
\end{aligned}
$$

From the above,

$$
\begin{aligned}
2\kappa\Big[\bar{\delta}\boldsymbol{\mathcal{L}}_{\rm G} + \partial_\mu(\boldsymbol{\mathcal{L}}_{\rm G}\xi^\mu)\Big] &= \xi^\mu(2\partial_\alpha \boldsymbol{G}^\alpha{}_\mu - \partial_\mu g_{\alpha\beta}\boldsymbol{G}^{\alpha\beta}) + \partial_\mu \boldsymbol{S}^\mu, \qquad (1.56) \\
\boldsymbol{S}^\mu &:= -2\xi^\lambda \boldsymbol{G}^\mu{}_\lambda + \frac{\partial \boldsymbol{G}}{\partial(\partial_\mu g_{\alpha\beta})}\bar{\delta}g_{\alpha\beta} + \frac{\partial \boldsymbol{D}^\mu}{\partial g_{\alpha\beta}}\bar{\delta}g_{\alpha\beta} + \frac{\partial \boldsymbol{D}^\mu}{\partial(\partial_\gamma g_{\alpha\beta})}\partial_\gamma \bar{\delta}g_{\alpha\beta} \\
&\quad + (\boldsymbol{G} + \partial_\alpha \boldsymbol{D}^\alpha)\xi^\mu. \qquad (1.57)
\end{aligned}
$$

Substituting this into (1.52),

$$
\begin{aligned}
2\kappa\delta S_{\rm G} &= \int_V d^D x\ \xi^\mu(2\partial_\alpha \boldsymbol{G}^\alpha{}_\mu - \partial_\mu g_{\alpha\beta}\boldsymbol{G}^{\alpha\beta}) \\
&\quad + \int_V d^D x\ \partial_\mu \boldsymbol{S}^\mu \equiv 0. \qquad (1.58)
\end{aligned}
$$

The second term becomes a surface integral over ∂V, and ξ^μ can be chosen so that it vanishes. Therefore, from the first term,

$$
\partial_\alpha \boldsymbol{G}^\alpha{}_\mu - \frac{1}{2}\partial_\mu g_{\alpha\beta}\boldsymbol{G}^{\alpha\beta} \equiv 0. \qquad (1.59)
$$

This is (1.17), namely the Bianchi identity.

Substituting (1.59) into (1.58), we obtain the identity

$$
\partial_\mu \boldsymbol{S}^\mu \equiv 0. \qquad (1.60)
$$

Now, we put

$$
\frac{1}{2\kappa}\boldsymbol{S}^\mu = \boldsymbol{B}^\mu{}_\gamma \xi^\gamma + \boldsymbol{C}^{\mu,\alpha}{}_\gamma \partial_\alpha \xi^\gamma + \boldsymbol{F}^{\mu,\alpha\beta}{}_\gamma \partial_\alpha \partial_\beta \xi^\gamma \qquad (1.61)
$$

and assume $\boldsymbol{F}^{\mu,\alpha\beta}{}_{\gamma} = \boldsymbol{F}^{\mu,\beta\alpha}{}_{\gamma}$. Then, by setting to zero the coefficients of ξ^γ, $\partial_\alpha \xi^\gamma$, $\partial_\alpha \partial_\beta \xi^\gamma$, and $\partial_\alpha \partial_\beta \partial_\gamma \xi^\mu$, respectively, we obtain

$$\partial_\mu \boldsymbol{B}^\mu{}_\gamma \equiv 0, \tag{1.62}$$
$$\boldsymbol{B}^\mu{}_\gamma + \partial_\alpha \boldsymbol{C}^{\alpha,\mu}{}_\gamma \equiv 0, \tag{1.63}$$
$$\boldsymbol{C}^{(\alpha,\beta)}{}_\gamma + \partial_\mu \boldsymbol{F}^{\mu,\alpha\beta}{}_\gamma \equiv 0, \tag{1.64}$$
$$\boldsymbol{F}^{(\gamma,\alpha\beta)}{}_\mu \equiv 0. \tag{1.65}$$

Here, $(\cdots)$ denotes symmetrization, and

$$\boldsymbol{C}^{(\alpha,\beta)}{}_\gamma = \frac{1}{2}(\boldsymbol{C}^{\alpha,\beta}{}_\gamma + \boldsymbol{C}^{\beta,\alpha}{}_\gamma), \tag{1.66}$$
$$\boldsymbol{F}^{(\gamma,\alpha\beta)}{}_\mu = \frac{1}{3}(\boldsymbol{F}^{\gamma,\alpha\beta}{}_\mu + \boldsymbol{F}^{\alpha,\beta\gamma}{}_\mu + \boldsymbol{F}^{\beta,\gamma\alpha}{}_\mu). \tag{1.67}$$

Because of

$$\begin{aligned}\boldsymbol{S}^\mu = \xi^\lambda(-2\boldsymbol{G}^\mu{}_\lambda + \boldsymbol{G}\delta^\mu_\lambda + \partial_\alpha \boldsymbol{D}^\alpha \delta^\mu_\lambda) - \Big(\frac{\partial \boldsymbol{G}}{\partial(\partial_\mu g_{\alpha\beta})} + \frac{\partial \boldsymbol{D}^\mu}{\partial g_{\alpha\beta}}\Big)(2\partial_\alpha \xi^\lambda g_{\lambda\beta} + \xi^\lambda \partial_\lambda g_{\alpha\beta}) \\ -\frac{\partial \boldsymbol{D}^\mu}{\partial(\partial_\gamma g_{\alpha\beta})}(2\partial_\gamma\partial_\alpha \xi^\lambda g_{\lambda\beta} + 2\partial_\alpha \xi^\lambda \partial_\gamma g_{\lambda\beta} + \partial_\gamma \xi^\lambda \partial_\lambda g_{\alpha\beta} + \xi^\lambda \partial_\gamma \partial_\lambda g_{\alpha\beta}),\end{aligned} \tag{1.68}$$

the quantities $\boldsymbol{B}^\mu{}_\gamma$, $\boldsymbol{C}^{\mu,\alpha}{}_\gamma$, and $\boldsymbol{F}^{\mu,\alpha\beta}{}_\gamma$ are as follows:

$$\boldsymbol{B}^\mu{}_\gamma = -\Big(\frac{1}{\kappa}\boldsymbol{G}^\mu{}_\gamma + \boldsymbol{t}^\mu{}_\gamma\Big) + \frac{1}{2\kappa}\partial_\sigma(\boldsymbol{D}^\sigma \delta^\mu_\gamma - \boldsymbol{D}^\mu \delta^\sigma_\gamma), \tag{1.69}$$
$$\boldsymbol{C}^{\mu,\alpha}{}_\gamma = -\frac{1}{\kappa}\Big(\frac{\partial \boldsymbol{G}}{\partial(\partial_\mu g_{\alpha\beta})} g_{\gamma\beta} + \frac{\partial \boldsymbol{D}^\mu}{\partial g_{\alpha\beta}} g_{\gamma\beta} + \frac{\partial \boldsymbol{D}^\mu}{\partial(\partial_\lambda g_{\alpha\beta})}\partial_\lambda g_{\gamma\beta} + \frac{1}{2}\frac{\partial \boldsymbol{D}^\mu}{\partial(\partial_\alpha g_{\mu\nu})}\partial_\gamma g_{\mu\nu}\Big), \tag{1.70}$$
$$\boldsymbol{F}^{\mu,\alpha\beta}{}_\gamma = -\frac{1}{2\kappa}\Big(\frac{\partial \boldsymbol{D}^\mu}{\partial(\partial_\beta g_{\alpha\delta})} + \frac{\partial \boldsymbol{D}^\mu}{\partial(\partial_\alpha g_{\beta\delta})}\Big) g_{\gamma\delta}. \tag{1.71}$$

For the last term of $\boldsymbol{B}^\mu{}_\gamma$, we used

$$\partial_\gamma \boldsymbol{D}^\mu = \frac{\partial \boldsymbol{D}^\mu}{\partial g_{\alpha\beta}}\partial_\gamma g_{\alpha\beta} + \frac{\partial \boldsymbol{D}^\mu}{\partial(\partial_\delta g_{\alpha\beta})}\partial_\gamma \partial_\delta g_{\alpha\beta}. \tag{1.72}$$

Note that $\boldsymbol{t}^\mu{}_\gamma$ is defined by (1.36). $\boldsymbol{t}^\mu{}_\gamma$ behaves as a tensor under the affine transformation (1.43).

From (1.62) and (1.69), we obtain

$$\partial_\mu\Big(\frac{1}{\kappa}\boldsymbol{G}^\mu{}_\gamma + \boldsymbol{t}^\mu{}_\gamma\Big) \equiv 0. \tag{1.73}$$

From this and the Einstein equation, we see that $\boldsymbol{t}^\mu{}_\gamma$ is an energy pseudotensor.

For $\tilde{S}_{\mathrm{G}}$, the same argument holds for any infinitesimal ξ^γ satisfying

$$\partial_\alpha \partial_\beta \xi^\gamma = 0 \tag{1.74}$$

and

$$\partial_\mu {}^{(0)}\boldsymbol{S}^\mu \equiv 0. \tag{1.75}$$

Here, ${}^{(0)}\boldsymbol{S}^\mu$ is obtained from $\boldsymbol{S}^\mu$ by setting $\boldsymbol{D}^\mu \to 0$. Therefore, if we set

$$\frac{1}{2\kappa} {}^{(0)}\boldsymbol{S}^\mu = {}^{(0)}\boldsymbol{B}^\mu{}_\gamma \xi^\gamma + {}^{(0)}\boldsymbol{C}^{\mu\alpha}{}_\gamma \partial_\alpha \xi^\gamma, \tag{1.76}$$

we obtain

$$\partial_\mu {}^{(0)}\boldsymbol{B}^\mu{}_\gamma \equiv 0, \tag{1.77}$$

$${}^{(0)}\boldsymbol{B}^\mu{}_\gamma + \partial_\alpha {}^{(0)}\boldsymbol{C}^{\alpha\mu}{}_\gamma \equiv 0. \tag{1.78}$$

${}^{(0)}\boldsymbol{B}^\mu{}_\gamma$ and ${}^{(0)}\boldsymbol{C}^{\mu,\alpha}{}_\gamma$ are obtained from $\boldsymbol{B}^\mu{}_\gamma$ and $\boldsymbol{C}^{\mu,\alpha}{}_\gamma$ by setting $\boldsymbol{D}^\mu \to 0$:

$${}^{(0)}\boldsymbol{B}^\mu{}_\gamma = -\Big(\frac{1}{\kappa}\boldsymbol{G}^\mu{}_\gamma + \boldsymbol{t}^\mu{}_\gamma\Big), \tag{1.79}$$

$${}^{(0)}\boldsymbol{C}^{\mu\alpha}{}_\gamma = -\frac{1}{\kappa}\frac{\partial \boldsymbol{G}}{\partial(\partial_\mu g_{\alpha\beta})} g_{\gamma\beta}. \tag{1.80}$$

From (1.78), (1.79), and (1.80), we see that ${}^{(0)}\boldsymbol{C}^{\mu\alpha}{}_\gamma$ is a pseudo-superpotential [2) 3)] .

Now, from (1.62), (1.63), (1.69), and (1.70), we obtain

$$\frac{1}{\kappa}\boldsymbol{G}^\mu{}_\nu + \boldsymbol{t}^\mu{}_\nu \equiv \partial_\lambda \Big[-\frac{1}{\kappa}\delta^{[\lambda}_\nu \boldsymbol{D}^{\mu]} + \boldsymbol{C}^{\lambda,\mu}{}_\nu \Big]. \tag{1.81}$$

Here, $[\cdots]$ denotes antisymmetrization, and

$$A^{[\mu\nu]} = \frac{1}{2}(A^{\mu\nu} - A^{\nu\mu}). \tag{1.82}$$

Here,

$$\begin{aligned} \boldsymbol{C}^{\lambda,\mu}{}_\nu &= \boldsymbol{C}^{[\lambda,\mu]}{}_\nu + \boldsymbol{C}^{(\lambda,\mu)}{}_\nu \\ &\equiv \boldsymbol{C}^{[\lambda,\mu]}{}_\nu - \partial_\rho \boldsymbol{F}^{\rho,\lambda\mu}{}_\nu \end{aligned} \tag{1.83}$$

where (1.64) has been used. Therefore,

$$\begin{aligned} \frac{1}{\kappa}\boldsymbol{G}^\mu{}_\nu + \boldsymbol{t}^\mu{}_\nu \equiv{}& \partial_\lambda \Big[-\frac{1}{\kappa}\delta^{[\lambda}_\nu \boldsymbol{D}^{\mu]} + \boldsymbol{C}^{[\lambda,\mu]}{}_\nu \Big] \\ & -\frac{1}{2}\partial_\lambda\partial_\rho(\boldsymbol{F}^{\rho,\lambda\mu}{}_\nu + \boldsymbol{F}^{\lambda,\rho\mu}{}_\nu). \end{aligned} \tag{1.84}$$

From (1.65),

$$\boldsymbol{F}^{\rho,\lambda\mu}{}_\nu + \boldsymbol{F}^{\lambda,\rho\mu}{}_\nu \equiv -\boldsymbol{F}^{\mu,\lambda\rho}{}_\nu. \tag{1.85}$$

[2)] Equation (1.39) follows from (1.73), (1.78), (1.79), and (1.80).

[3)] Einstein's energy pseudotensor and the corresponding pseudo-superpotential were discovered by Einstein in 1916.

Therefore,

$$\begin{aligned}\boldsymbol{F}^{\rho,\lambda\mu}{}_{\nu} + \boldsymbol{F}^{\lambda,\rho\mu}{}_{\nu} &\equiv \frac{1}{3}\Big[\boldsymbol{F}^{\rho,\lambda\mu}{}_{\nu} + \boldsymbol{F}^{\lambda,\rho\mu}{}_{\nu} + 2(-\boldsymbol{F}^{\mu,\lambda\rho}{}_{\nu})\Big] \\ &= \frac{1}{3}\Big[(\boldsymbol{F}^{\rho,\lambda\mu}{}_{\nu} - \boldsymbol{F}^{\mu,\lambda\rho}{}_{\nu}) + (\boldsymbol{F}^{\lambda,\rho\mu}{}_{\nu} - \boldsymbol{F}^{\mu,\rho\lambda}{}_{\nu})\Big]\end{aligned} \tag{1.86}$$

and we obtain

$$-\frac{1}{2}\partial_\lambda\partial_\rho(\boldsymbol{F}^{\rho,\lambda\mu}{}_{\nu} + \boldsymbol{F}^{\lambda,\rho\mu}{}_{\nu}) \;\equiv\; \partial_\lambda \boldsymbol{u}^{\lambda\mu}{}_{\nu}, \tag{1.87}$$

$$\boldsymbol{u}^{\lambda\mu}{}_{\nu} := \partial_\rho \boldsymbol{v}^{\rho\lambda\mu}{}_{\nu} \equiv \boldsymbol{u}^{[\lambda\mu]}{}_{\nu}, \tag{1.88}$$

$$\boldsymbol{v}^{\rho\lambda\mu}{}_{\nu} := -\frac{1}{3}(\boldsymbol{F}^{\lambda,\rho\mu}{}_{\nu} - \boldsymbol{F}^{\mu,\rho\lambda}{}_{\nu}) = \boldsymbol{v}^{\rho[\lambda\mu]}{}_{\nu}. \tag{1.89}$$

From the above,

$$\frac{1}{\kappa}\boldsymbol{G}^{\mu}{}_{\nu} + \boldsymbol{t}^{\mu}{}_{\nu} \;\equiv\; \partial_\lambda \boldsymbol{U}^{\lambda\mu}{}_{\nu}, \tag{1.90}$$

$$\boldsymbol{U}^{\lambda\mu}{}_{\nu} := -\frac{1}{\kappa}\delta^{[\lambda}_{\nu}\boldsymbol{D}^{\mu]} + \boldsymbol{C}^{[\lambda,\mu]}{}_{\nu} - \frac{1}{3}\partial_\rho(\boldsymbol{F}^{\lambda,\rho\mu}{}_{\nu} - \boldsymbol{F}^{\mu,\rho\lambda}{}_{\nu}) = \boldsymbol{U}^{[\lambda\mu]}{}_{\nu}. \tag{1.91}$$

This $\boldsymbol{U}^{\lambda\mu}{}_{\nu}$ is a superpotential (in this article, we call it Utiyama's superpotential).

2 Explicit Computations

2.1 Computation of the Quantities Appearing in the Previous Section

The notation in the previous section was

$$\boldsymbol{G} = \sqrt{-g}G, \quad G = g^{\mu\nu}\Big[\Gamma^{\rho}{}_{\gamma\nu}\Gamma^{\gamma}{}_{\mu\rho} - \Gamma^{\rho}{}_{\gamma\rho}\Gamma^{\gamma}{}_{\mu\nu}\Big], \tag{2.1}$$

$$\boldsymbol{D}^{\rho} = \sqrt{-g}D^{\rho}, \quad D^{\rho} = g^{\mu\nu}\Gamma^{\rho}{}_{\mu\nu} - g^{\mu\rho}\Gamma^{\nu}{}_{\mu\nu}, \tag{2.2}$$

$$\boldsymbol{B}^{\mu}{}_{\gamma} = -\Big(\frac{1}{\kappa}\boldsymbol{G}^{\mu}{}_{\gamma} + \boldsymbol{t}^{\mu}{}_{\gamma}\Big) + \frac{1}{2\kappa}\partial_\sigma(\boldsymbol{D}^{\sigma}\delta^{\mu}_{\gamma} - \boldsymbol{D}^{\mu}\delta^{\sigma}_{\gamma}), \tag{2.3}$$

$$\boldsymbol{C}^{\mu,\alpha}{}_{\gamma} = -\frac{1}{\kappa}\Big(\frac{\partial \boldsymbol{G}}{\partial(\partial_\mu g_{\alpha\beta})}g_{\gamma\beta} + \frac{\partial \boldsymbol{D}^{\mu}}{\partial g_{\alpha\beta}}g_{\gamma\beta} + \frac{\partial \boldsymbol{D}^{\mu}}{\partial(\partial_\lambda g_{\alpha\beta})}\partial_\lambda g_{\gamma\beta} + \frac{1}{2}\frac{\partial \boldsymbol{D}^{\mu}}{\partial(\partial_\alpha g_{\mu\nu})}\partial_\gamma g_{\mu\nu}\Big), \tag{2.4}$$

$$\boldsymbol{F}^{\mu,\alpha\beta}{}_{\gamma} = -\frac{1}{2\kappa}\Big(\frac{\partial \boldsymbol{D}^{\mu}}{\partial(\partial_\beta g_{\alpha\delta})} + \frac{\partial \boldsymbol{D}^{\mu}}{\partial(\partial_\alpha g_{\beta\delta})}\Big)g_{\gamma\delta}, \tag{2.5}$$

$${}^{(0)}\boldsymbol{C}^{\mu\alpha}{}_{\gamma} = -\frac{1}{\kappa}\frac{\partial \boldsymbol{G}}{\partial(\partial_\mu g_{\alpha\beta})}g_{\gamma\beta}, \tag{2.6}$$

$$\boldsymbol{t}^{\mu}{}_{\nu} = \frac{1}{2\kappa}\Big(\frac{\partial \boldsymbol{G}}{\partial(\partial_\mu g_{\alpha\beta})}\partial_\nu g_{\alpha\beta} - \delta^{\mu}_{\nu}\boldsymbol{G}\Big). \tag{2.7}$$

We now compute these explicitly.

Now, we put

$$\Gamma_{\lambda\mu\nu} := \frac{1}{2}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}), \tag{2.8}$$

then

$$
\begin{aligned}
G &= g^{\mu\nu} g^{\rho\alpha} g^{\gamma\beta} [\Gamma_{\alpha\gamma\nu}\Gamma_{\beta\mu\rho} - \Gamma_{\alpha\gamma\rho}\Gamma_{\beta\mu\nu}] \\
&= g^{\mu\nu} g^{\rho\alpha} g^{\gamma\beta} [\Gamma_{\alpha\gamma\nu}\Gamma_{\beta\mu\rho} - \frac{1}{2}\partial_\gamma g_{\alpha\rho}\Gamma_{\beta\mu\nu}]. \qquad (2.9)
\end{aligned}
$$

Also,

$$
\frac{\partial \Gamma_{\lambda\mu\nu}}{\partial(\partial_\sigma g_{\delta\tau})} = \frac{1}{2}(\delta^\sigma_\mu \delta^{\delta\tau}_{\lambda\nu} + \delta^\sigma_\nu \delta^{\delta\tau}_{\lambda\mu} - \delta^\sigma_\lambda \delta^{\delta\tau}_{\mu\nu}) \qquad (2.10)
$$

where

$$
\delta^{\alpha\beta}_{\mu\nu} = \delta^{(\alpha}_\mu \delta^{\beta)}_\nu. \qquad (2.11)
$$

Therefore,

$$
\begin{aligned}
G^{\sigma,\delta\tau} &:= \frac{\partial G}{\partial(\partial_\sigma g_{\delta\tau})} \\
&= \frac{1}{2} g^{\mu\nu} g^{\rho\alpha} g^{\gamma\beta} [(\delta^\sigma_\gamma \delta^{\delta\tau}_{\alpha\nu} + \delta^\sigma_\nu \delta^{\delta\tau}_{\alpha\gamma} - \delta^\sigma_\alpha \delta^{\delta\tau}_{\gamma\nu})\Gamma_{\beta\mu\rho} + \Gamma_{\alpha\gamma\nu}(\delta^\sigma_\mu \delta^{\delta\tau}_{\beta\rho} + \delta^\sigma_\rho \delta^{\delta\tau}_{\beta\mu} - \delta^\sigma_\beta \delta^{\delta\tau}_{\mu\rho}) \\
&\quad - \delta^\sigma_\gamma \delta^{\delta\tau}_{\alpha\rho}\Gamma_{\beta\mu\nu} - \Gamma_{\alpha\gamma\rho}(\delta^\sigma_\mu \delta^{\delta\tau}_{\beta\nu} + \delta^\sigma_\nu \delta^{\delta\tau}_{\beta\mu} - \delta^\sigma_\beta \delta^{\delta\tau}_{\mu\nu})]. \qquad (2.12)
\end{aligned}
$$

From this,

$$
\begin{aligned}
G^{\sigma,\delta\tau} &= \frac{1}{2}[g^{\mu\nu} g^{\rho\alpha}(\delta^\sigma_\gamma \delta^{\delta\tau}_{\alpha\nu} + \delta^\sigma_\nu \delta^{\delta\tau}_{\alpha\gamma} - \delta^\sigma_\alpha \delta^{\delta\tau}_{\gamma\nu})\Gamma^\gamma{}_{\mu\rho} + g^{\mu\nu} g^{\gamma\beta}\Gamma^\rho{}_{\gamma\nu}(\delta^\sigma_\mu \delta^{\delta\tau}_{\beta\rho} + \delta^\sigma_\rho \delta^{\delta\tau}_{\beta\mu} - \delta^\sigma_\beta \delta^{\delta\tau}_{\mu\rho}) \\
&\quad - g^{\mu\nu} g^{\rho\alpha} \delta^\sigma_\gamma \delta^{\delta\tau}_{\alpha\rho}\Gamma^\gamma{}_{\mu\nu} - g^{\mu\nu} g^{\gamma\beta}\Gamma_\gamma(\delta^\sigma_\mu \delta^{\delta\tau}_{\beta\nu} + \delta^\sigma_\nu \delta^{\delta\tau}_{\beta\mu} - \delta^\sigma_\beta \delta^{\delta\tau}_{\mu\nu})] \\
&=: (1) + (2) + (3) + (4). \qquad (2.13)
\end{aligned}
$$

Here,

$$
\Gamma_\gamma := \Gamma^\rho{}_{\gamma\rho}. \qquad (2.14)
$$

First,

$$
\begin{aligned}
(1) &= \frac{1}{2} g^{\mu\nu} g^{\rho\alpha}(\delta^\sigma_\gamma \delta^{\delta\tau}_{\alpha\nu} + \delta^\sigma_\nu \delta^{\delta\tau}_{\alpha\gamma} - \delta^\sigma_\alpha \delta^{\delta\tau}_{\gamma\nu})\Gamma^\gamma{}_{\mu\rho} \\
&= \frac{1}{4}[\Gamma^\sigma{}_{\mu\rho}(g^{\mu\tau} g^{\rho\delta} + g^{\mu\delta} g^{\rho\tau}) + g^{\mu\sigma}(g^{\rho\delta}\Gamma^\tau{}_{\mu\rho} + g^{\rho\tau}\Gamma^\delta{}_{\mu\rho}) - g^{\rho\sigma}(g^{\mu\tau}\Gamma^\delta{}_{\mu\rho} + g^{\mu\delta}\Gamma^\tau{}_{\mu\rho})] \\
&= \frac{1}{4}\Gamma^\sigma{}_{\mu\rho}(g^{\mu\tau} g^{\rho\delta} + g^{\mu\delta} g^{\rho\tau}). \qquad (2.15)
\end{aligned}
$$

Next,

$$
\begin{aligned}
(2) &= \frac{1}{2} g^{\mu\nu} g^{\gamma\beta}\Gamma^\rho{}_{\gamma\nu}(\delta^\sigma_\mu \delta^{\delta\tau}_{\beta\rho} + \delta^\sigma_\rho \delta^{\delta\tau}_{\beta\mu} - \delta^\sigma_\beta \delta^{\delta\tau}_{\mu\rho}) \\
&= \frac{1}{4}[g^{\sigma\nu}(g^{\gamma\delta}\Gamma^\tau{}_{\gamma\nu} + g^{\gamma\tau}\Gamma^\delta{}_{\gamma\nu}) + \Gamma^\sigma{}_{\gamma\nu}(g^{\tau\nu} g^{\gamma\delta} + g^{\delta\nu} g^{\gamma\tau}) - g^{\gamma\sigma}(g^{\delta\nu}\Gamma^\tau{}_{\gamma\nu} + g^{\tau\nu}\Gamma^\delta{}_{\gamma\nu})] \\
&= \frac{1}{4}\Gamma^\sigma{}_{\gamma\nu}(g^{\tau\nu} g^{\gamma\delta} + g^{\delta\nu} g^{\gamma\tau}) \qquad (2.16)
\end{aligned}
$$

and

$$\begin{aligned}
(3) &= -\frac{1}{2} g^{\mu\nu} g^{\rho\alpha} \delta^{\sigma}_{\gamma} \delta^{\delta\tau}_{\alpha\rho} \Gamma^{\gamma}{}_{\mu\nu} \\
&= \frac{1}{4}(-2 g^{\mu\nu} g^{\delta\tau} \Gamma^{\sigma}{}_{\mu\nu}), \qquad (2.17) \\
(4) &= -\frac{1}{2} g^{\mu\nu} g^{\gamma\beta} \Gamma_{\gamma} (\delta^{\sigma}_{\mu} \delta^{\delta\tau}_{\beta\nu} + \delta^{\sigma}_{\nu} \delta^{\delta\tau}_{\beta\mu} - \delta^{\sigma}_{\beta} \delta^{\delta\tau}_{\mu\nu}) \\
&= \frac{1}{4}[-2(g^{\sigma\tau} g^{\gamma\delta} + g^{\sigma\delta} g^{\gamma\tau})\Gamma_{\gamma} + 2 g^{\delta\tau} g^{\sigma\gamma} \Gamma_{\gamma}] \qquad (2.18)
\end{aligned}$$

are obtained. Therefore,

$$\begin{aligned}
G^{\sigma,\delta\tau} &= \frac{1}{4}\Big[2\Gamma^{\sigma}{}_{\mu\rho}(g^{\mu\tau} g^{\rho\delta} + g^{\mu\delta} g^{\rho\tau}) - 2 g^{\mu\nu} g^{\delta\tau} \Gamma^{\sigma}{}_{\mu\nu} - 2(g^{\sigma\tau} g^{\gamma\delta} + g^{\sigma\delta} g^{\gamma\tau})\Gamma_{\gamma} + 2 g^{\delta\tau} g^{\sigma\gamma} \Gamma_{\gamma}\Big] \\
&= \frac{1}{2}\Big[\Gamma^{\sigma}{}_{\mu\rho}(2 g^{\mu\tau} g^{\rho\delta} - g^{\mu\rho} g^{\delta\tau}) + (g^{\delta\tau} g^{\sigma\gamma} - g^{\sigma\tau} g^{\gamma\delta} - g^{\sigma\delta} g^{\gamma\tau})\Gamma_{\gamma}\Big]. \qquad (2.19)
\end{aligned}$$

From this,

$$\begin{aligned}
{}^{(0)}\boldsymbol{C}^{\sigma\delta}{}_{\alpha} &= -\frac{1}{\kappa}\sqrt{-g}\, G^{\sigma,\delta\tau} g_{\tau\alpha} \\
&= -\frac{\sqrt{-g}}{2\kappa}\Big[\Gamma^{\sigma}{}_{\mu\rho}(2\delta^{\mu}_{\alpha} g^{\rho\delta} - g^{\mu\rho}\delta^{\delta}_{\alpha}) + (\delta^{\delta}_{\alpha} g^{\sigma\gamma} - \delta^{\sigma}_{\alpha} g^{\gamma\delta} - g^{\sigma\delta}\delta^{\gamma}_{\alpha})\Gamma_{\gamma}\Big] \\
&= -\frac{\sqrt{-g}}{2\kappa}\Big[(2\Gamma^{\sigma}{}_{\alpha\rho} g^{\rho\delta} - \Gamma^{\sigma}{}_{\mu\rho} g^{\mu\rho}\delta^{\delta}_{\alpha}) + (\delta^{\delta}_{\alpha} g^{\sigma\gamma}\Gamma_{\gamma} - \delta^{\sigma}_{\alpha} g^{\gamma\delta}\Gamma_{\gamma} - g^{\sigma\delta}\Gamma_{\alpha})\Big] \\
&= \frac{1}{2\kappa}\sqrt{-g}\Big[\delta^{\delta}_{\alpha}(\Gamma^{\sigma}{}_{\mu\rho} g^{\mu\rho} - \Gamma_{\gamma} g^{\sigma\gamma}) + \delta^{\sigma}_{\alpha}\Gamma_{\gamma} g^{\gamma\delta} + \Gamma_{\alpha} g^{\sigma\delta} - 2\Gamma^{\sigma}{}_{\alpha\rho} g^{\rho\delta}\Big]. \qquad (2.20)
\end{aligned}$$

Next, we put

$$D^{\mu\sigma,\alpha\beta} := \frac{\partial D^{\mu}}{\partial(\partial_{\sigma} g_{\alpha\beta})}. \qquad (2.21)$$

We examine this. First,

$$D^{\kappa} = (g^{\mu\nu} g^{\kappa\rho} - g^{\mu\kappa} g^{\nu\rho})\Gamma_{\rho\mu\nu}. \qquad (2.22)$$

Therefore,

$$\begin{aligned}
D^{\kappa\sigma,\alpha\beta} &= \frac{1}{2}(g^{\mu\nu} g^{\kappa\rho} - g^{\mu\kappa} g^{\nu\rho})(\delta^{\sigma}_{\mu}\delta^{\alpha\beta}_{\rho\nu} + \delta^{\sigma}_{\nu}\delta^{\alpha\beta}_{\rho\mu} - \delta^{\sigma}_{\rho}\delta^{\alpha\beta}_{\mu\nu}) \\
&= \frac{1}{4}(2 g^{\sigma\beta} g^{\kappa\alpha} + 2 g^{\sigma\alpha} g^{\kappa\beta} - 2 g^{\alpha\beta} g^{\kappa\sigma} \\
&\qquad -2 g^{\sigma\kappa} g^{\beta\alpha} - g^{\beta\kappa} g^{\sigma\alpha} - g^{\alpha\kappa} g^{\sigma\beta} + g^{\alpha\kappa} g^{\beta\sigma} + g^{\beta\kappa} g^{\alpha\sigma}) \\
&= \frac{1}{2}(g^{\sigma\beta} g^{\kappa\alpha} + g^{\sigma\alpha} g^{\kappa\beta} - 2 g^{\alpha\beta} g^{\kappa\sigma}). \qquad (2.23)
\end{aligned}$$

From this,

$$
\begin{aligned}
\boldsymbol{F}^{\kappa,\alpha\beta}{}_{\gamma} &= -\frac{\sqrt{-g}}{2\kappa}\Big(D^{\kappa\beta,\alpha\delta}+D^{\kappa\alpha,\beta\delta}\Big)g_{\gamma\delta} \\
&= \frac{\sqrt{-g}}{4\kappa}\Big(-g^{\beta\delta}g^{\kappa\alpha}-g^{\beta\alpha}g^{\kappa\delta}+2g^{\alpha\delta}g^{\kappa\beta}-g^{\alpha\delta}g^{\kappa\beta}-g^{\alpha\beta}g^{\kappa\delta}+2g^{\beta\delta}g^{\kappa\alpha}\Big)g_{\gamma\delta} \\
&= \frac{\sqrt{-g}}{4\kappa}\Big(-2g^{\alpha\beta}g^{\kappa\delta}+g^{\alpha\delta}g^{\kappa\beta}+g^{\beta\delta}g^{\kappa\alpha}\Big)g_{\gamma\delta} \\
&= \frac{\sqrt{-g}}{4\kappa}\Big(-2\delta^{\kappa}_{\gamma}g^{\alpha\beta}+\delta^{\alpha}_{\gamma}g^{\kappa\beta}+\delta^{\beta}_{\gamma}g^{\kappa\alpha}\Big).
\end{aligned}
\tag{2.24}
$$

Next, we examine $\boldsymbol{C}^{\mu,\alpha}{}_{\gamma}$. Now, we put

$$ {}^{(1)}\boldsymbol{c}^{\mu,\alpha}{}_{\gamma} := \frac{\partial \boldsymbol{D}^{\mu}}{\partial g_{\alpha\beta}}g_{\gamma\beta}, \tag{2.25} $$

$$ {}^{(2)}\boldsymbol{c}^{\mu,\alpha}{}_{\gamma} := \frac{\partial \boldsymbol{D}^{\mu}}{\partial(\partial_{\lambda} g_{\alpha\beta})}\partial_{\lambda}g_{\gamma\beta}, \tag{2.26} $$

$$ {}^{(3)}\boldsymbol{c}^{\mu,\alpha}{}_{\gamma} := \frac{1}{2}\frac{\partial \boldsymbol{D}^{\mu}}{\partial(\partial_{\alpha} g_{\mu\nu})}\partial_{\gamma}g_{\mu\nu}, \tag{2.27} $$

$$ {}^{(0)}\boldsymbol{c}^{\mu,\alpha}{}_{\gamma} := -\kappa\cdot {}^{(0)}\boldsymbol{C}^{\mu\alpha}{}_{\gamma} = \frac{\partial \boldsymbol{G}}{\partial(\partial_{\mu} g_{\alpha\beta})}g_{\gamma\beta}, \tag{2.28} $$

then

$$ \boldsymbol{C}^{\mu,\alpha}{}_{\gamma} = -\frac{1}{\kappa}\sum_{n=0}^{3}{}^{(n)}\boldsymbol{c}^{\mu,\alpha}{}_{\gamma}. \tag{2.29} $$

First, we examine ${}^{(1)}\boldsymbol{c}^{\mu,\alpha}{}_{\gamma}$. Now, we put

$$ d^{\kappa,\alpha\beta} := \frac{\partial D^{\kappa}}{\partial g_{\alpha\beta}}. \tag{2.30} $$

From the relation

$$ \delta g^{\sigma\lambda} = -g^{\sigma(\alpha}g^{\beta)\lambda}\delta g_{\alpha\beta}, \tag{2.31} $$

it follows that

$$
\begin{aligned}
d^{\kappa,\alpha\beta} &= (-g^{\mu(\alpha}g^{\beta)\nu}g^{\kappa\rho}-g^{\mu\nu}g^{\kappa(\alpha}g^{\beta)\rho}+g^{\mu(\alpha}g^{\beta)\kappa}g^{\nu\rho}+g^{\mu\kappa}g^{\nu(\alpha}g^{\beta)\rho})\Gamma_{\rho\mu\nu} \\
&= -g^{\mu\alpha}g^{\beta\nu}\Gamma^{\kappa}{}_{\mu\nu}-\frac{1}{2}g^{\mu\nu}(g^{\alpha\kappa}\Gamma^{\beta}{}_{\mu\nu}+g^{\beta\kappa}\Gamma^{\alpha}{}_{\mu\nu}) \\
&\quad +\frac{1}{2}(g^{\mu\alpha}g^{\beta\kappa}+g^{\mu\beta}g^{\alpha\kappa})\Gamma_{\mu}+\frac{1}{2}g^{\mu\kappa}(g^{\nu\alpha}\Gamma^{\beta}{}_{\mu\nu}+g^{\nu\beta}\Gamma^{\alpha}{}_{\mu\nu}).
\end{aligned}
\tag{2.32}
$$

Therefore,

$$
\begin{aligned}
\frac{\partial \boldsymbol{D}^{\kappa}}{\partial g_{\alpha\beta}} &= \frac{1}{2}g^{\alpha\beta}\boldsymbol{D}^{\kappa}+\sqrt{-g}d^{\kappa,\alpha\beta} \\
&= \frac{\sqrt{-g}}{2}\Big[g^{\alpha\beta}(g^{\mu\nu}\Gamma^{\kappa}{}_{\mu\nu}-g^{\mu\kappa}\Gamma_{\mu})-2g^{\mu\alpha}g^{\beta\nu}\Gamma^{\kappa}{}_{\mu\nu}-g^{\mu\nu}(g^{\alpha\kappa}\Gamma^{\beta}{}_{\mu\nu}+g^{\beta\kappa}\Gamma^{\alpha}{}_{\mu\nu}) \\
&\quad +(g^{\mu\alpha}g^{\beta\kappa}+g^{\mu\beta}g^{\alpha\kappa})\Gamma_{\mu}+g^{\mu\kappa}(g^{\nu\alpha}\Gamma^{\beta}{}_{\mu\nu}+g^{\nu\beta}\Gamma^{\alpha}{}_{\mu\nu})\Big].
\end{aligned}
\tag{2.33}
$$

From this,

$$
\begin{aligned}
{}^{(1)}\boldsymbol{c}^{\kappa,\alpha}{}_{\gamma} &= \frac{\partial \boldsymbol{D}^{\kappa}}{\partial g_{\alpha\beta}} g_{\gamma\beta} \\
&= \frac{\sqrt{-g}}{2}\Big[\delta^{\alpha}_{\gamma}(g^{\mu\nu}\Gamma^{\kappa}{}_{\mu\nu} - g^{\mu\kappa}\Gamma_{\mu}) - 2g^{\mu\alpha}\Gamma^{\kappa}{}_{\mu\gamma} - g^{\mu\nu}(g^{\alpha\kappa}g_{\beta\gamma}\Gamma^{\beta}{}_{\mu\nu} + \delta^{\kappa}_{\gamma}\Gamma^{\alpha}{}_{\mu\nu}) \\
&\quad +(g^{\mu\alpha}\delta^{\kappa}_{\gamma}\Gamma_{\mu} + g^{\alpha\kappa}\Gamma_{\gamma}) + g^{\mu\kappa}(g^{\nu\alpha}g_{\beta\gamma}\Gamma^{\beta}{}_{\mu\nu} + \Gamma^{\alpha}{}_{\mu\gamma})\Big].
\end{aligned} \tag{2.34}
$$

Next, we examine ${}^{(2)}\boldsymbol{c}^{\mu,\alpha}{}_{\gamma}$ and ${}^{(3)}\boldsymbol{c}^{\mu,\alpha}{}_{\gamma}$. From the relation

$$
\partial_{\lambda} g_{\mu\nu} = g_{\mu\rho}\Gamma^{\rho}{}_{\nu\lambda} + g_{\nu\rho}\Gamma^{\rho}{}_{\lambda\mu}, \tag{2.35}
$$

it follows that

$$
\begin{aligned}
{}^{(2)}\boldsymbol{c}^{\kappa,\alpha}{}_{\gamma} &= \frac{\partial \boldsymbol{D}^{\kappa}}{\partial(\partial_{\lambda} g_{\alpha\beta})}\partial_{\lambda} g_{\gamma\beta} \\
&= \sqrt{-g}D^{\kappa\lambda,\alpha\beta}(g_{\gamma\alpha}\Gamma^{\alpha}{}_{\beta\lambda} + g_{\beta\alpha}\Gamma^{\alpha}{}_{\lambda\gamma}) \\
&= \frac{\sqrt{-g}}{2}(g^{\lambda\beta}g^{\kappa\alpha} + g^{\lambda\alpha}g^{\kappa\beta} - 2g^{\alpha\beta}g^{\kappa\lambda})(g_{\gamma\rho}\Gamma^{\rho}{}_{\beta\lambda} + g_{\beta\rho}\Gamma^{\rho}{}_{\lambda\gamma}) \\
&= \frac{\sqrt{-g}}{2}(g^{\lambda\beta}g^{\kappa\alpha}g_{\gamma\rho}\Gamma^{\rho}{}_{\beta\lambda} + g^{\kappa\alpha}\Gamma_{\gamma} + g^{\lambda\alpha}g^{\kappa\beta}g_{\gamma\rho}\Gamma^{\rho}{}_{\beta\lambda} + g^{\lambda\alpha}\Gamma^{\kappa}{}_{\lambda\gamma} \\
&\quad -2g^{\alpha\beta}g^{\kappa\lambda}g_{\gamma\rho}\Gamma^{\rho}{}_{\beta\lambda} - 2g^{\kappa\lambda}\Gamma^{\alpha}{}_{\lambda\gamma}).
\end{aligned} \tag{2.36}
$$

Also,

$$
\begin{aligned}
{}^{(3)}\boldsymbol{c}^{\kappa,\alpha}{}_{\gamma} &:= \frac{1}{2}\frac{\partial \boldsymbol{D}^{\kappa}}{\partial(\partial_{\alpha} g_{\mu\nu})}\partial_{\gamma} g_{\mu\nu} \\
&= \frac{\sqrt{-g}}{2}\boldsymbol{D}^{\kappa\alpha,\mu\nu}\partial_{\gamma} g_{\mu\nu} \\
&= \frac{\sqrt{-g}}{2}\boldsymbol{D}^{\kappa\alpha,\mu\nu}(g_{\mu\rho}\Gamma^{\rho}{}_{\nu\gamma} + g_{\nu\rho}\Gamma^{\rho}{}_{\gamma\mu}) \\
&= \frac{\sqrt{-g}}{2}(g^{\alpha\nu}g^{\kappa\mu} + g^{\alpha\mu}g^{\kappa\nu} - 2g^{\mu\nu}g^{\kappa\alpha})g_{\mu\rho}\Gamma^{\rho}{}_{\nu\gamma} \\
&= \frac{\sqrt{-g}}{2}(g^{\alpha\nu}\Gamma^{\kappa}{}_{\nu\gamma} + g^{\kappa\nu}\Gamma^{\alpha}{}_{\nu\gamma} - 2g^{\kappa\alpha}\Gamma_{\gamma})
\end{aligned} \tag{2.37}
$$

holds. Note that

$$
{}^{(0)}\boldsymbol{c}^{\kappa,\alpha}{}_{\gamma} = \frac{\sqrt{-g}}{2}\Big[\delta^{\alpha}_{\gamma}(-g^{\mu\rho}\Gamma^{\kappa}{}_{\mu\rho} + g^{\kappa\lambda}\Gamma_{\lambda}) - \delta^{\kappa}_{\gamma}g^{\lambda\alpha}\Gamma_{\lambda} - g^{\kappa\alpha}\Gamma_{\gamma} + 2g^{\rho\alpha}\Gamma^{\kappa}{}_{\gamma\rho}\Big]. \tag{2.38}
$$

Therefore,

$$
\begin{aligned}
-\kappa \boldsymbol{C}^{\kappa,\alpha}{}_{\gamma} &= \sum_{n=0}^{3} {}^{(n)}\boldsymbol{c}^{\kappa,\alpha}{}_{\gamma} \\
&= \frac{\sqrt{-g}}{2}\Big[\delta^{\alpha}_{\gamma}(-g^{\mu\rho}\Gamma^{\kappa}{}_{\mu\rho} + g^{\kappa\lambda}\Gamma_{\lambda}) - \delta^{\kappa}_{\gamma}g^{\lambda\alpha}\Gamma_{\lambda} - g^{\kappa\alpha}\Gamma_{\gamma} + 2g^{\rho\alpha}\Gamma^{\kappa}{}_{\gamma\rho} \\
&\quad +\delta^{\alpha}_{\gamma}(g^{\mu\nu}\Gamma^{\kappa}{}_{\mu\nu} - g^{\mu\kappa}\Gamma_{\mu}) - 2g^{\mu\alpha}\Gamma^{\kappa}{}_{\mu\gamma} - g^{\mu\nu}(g^{\alpha\kappa}g_{\beta\gamma}\Gamma^{\beta}{}_{\mu\nu} + \delta^{\kappa}_{\gamma}\Gamma^{\alpha}{}_{\mu\nu}) \\
&\quad +(g^{\mu\alpha}\delta^{\kappa}_{\gamma}\Gamma_{\mu} + g^{\alpha\kappa}\Gamma_{\gamma}) + g^{\mu\kappa}(g^{\nu\alpha}g_{\beta\gamma}\Gamma^{\beta}{}_{\mu\nu} + \Gamma^{\alpha}{}_{\mu\gamma}) \\
&\quad +g^{\lambda\beta}g^{\kappa\alpha}g_{\gamma\rho}\Gamma^{\rho}{}_{\beta\lambda} + g^{\kappa\alpha}\Gamma_{\gamma} + g^{\lambda\alpha}g^{\kappa\beta}g_{\gamma\rho}\Gamma^{\rho}{}_{\beta\lambda} + g^{\lambda\alpha}\Gamma^{\kappa}{}_{\lambda\gamma} \\
&\quad -2g^{\alpha\beta}g^{\kappa\lambda}g_{\gamma\rho}\Gamma^{\rho}{}_{\beta\lambda} - 2g^{\kappa\lambda}\Gamma^{\alpha}{}_{\lambda\gamma} \\
&\quad +g^{\alpha\nu}\Gamma^{\kappa}{}_{\nu\gamma} + g^{\kappa\nu}\Gamma^{\alpha}{}_{\nu\gamma} - 2g^{\kappa\alpha}\Gamma_{\gamma}\Big].
\end{aligned} \tag{2.39}
$$

After simplification, we obtain

$$
\boldsymbol{C}^{\kappa,\alpha}{}_{\gamma} = \frac{\sqrt{-g}}{2\kappa}\Big[\delta^{\kappa}_{\gamma}g^{\mu\nu}\Gamma^{\alpha}{}_{\mu\nu} + g^{\kappa\alpha}\Gamma_{\gamma} - 2g^{\rho\alpha}\Gamma^{\kappa}{}_{\gamma\rho}\Big]. \tag{2.40}
$$

2.2 Named Superpotentials

Now,

$$
\boldsymbol{W}^{\kappa\rho,\alpha}{}_{\gamma} := \frac{\sqrt{-g}}{2\kappa}(g^{\alpha\kappa}\delta^{\rho}_{\gamma} - g^{\alpha\rho}\delta^{\kappa}_{\gamma}) = \frac{4}{3}\boldsymbol{F}^{[\kappa,\rho]\alpha}{}_{\gamma} = \boldsymbol{W}^{[\kappa\rho],\alpha}{}_{\gamma} \tag{2.41}
$$

and set

$$
\boldsymbol{f}^{\kappa\rho}{}_{\gamma} := {}^{(0)}\boldsymbol{C}^{\kappa\rho}{}_{\gamma} - \partial_{\rho}\boldsymbol{W}^{\kappa\rho,\alpha}{}_{\gamma} = \boldsymbol{f}^{[\kappa\rho]}{}_{\gamma}, \tag{2.42}
$$

$$
\boldsymbol{m}^{\kappa\rho}{}_{\gamma} := \boldsymbol{C}^{\kappa,\rho}{}_{\gamma} - \partial_{\rho}\boldsymbol{W}^{\kappa\rho,\alpha}{}_{\gamma} = \boldsymbol{m}^{[\kappa\rho]}{}_{\gamma}. \tag{2.43}
$$

We set these. $\boldsymbol{f}^{\kappa\rho}{}_{\gamma}$ is the Freud superpotential, and $\boldsymbol{m}^{\kappa\rho}{}_{\gamma}$ is the Møller superpotential [5]. They satisfy

$$
\partial_{\kappa}\boldsymbol{f}^{\kappa\rho}{}_{\gamma} = \partial_{\kappa}{}^{(0)}\boldsymbol{C}^{\kappa\rho}{}_{\gamma}, \tag{2.44}
$$

$$
\partial_{\kappa}\boldsymbol{m}^{\kappa\rho}{}_{\gamma} = \partial_{\kappa}\boldsymbol{C}^{\kappa,\rho}{}_{\gamma}. \tag{2.45}
$$

The Freud superpotential gives the Einstein energy pseudotensor, while the Møller superpotential gives the energy pseudotensor

$$
{}^{(M)}\boldsymbol{t}^{\mu}{}_{\nu} := \boldsymbol{t}^{\mu}{}_{\nu} + \frac{1}{\kappa}\partial_{\lambda}\delta^{[\lambda}_{\nu}\boldsymbol{D}^{\mu]}. \tag{2.46}
$$

Now let us find the expression for $\boldsymbol{f}^{\kappa\rho}{}_{\gamma}$. First,

$$
\partial_{\rho}\sqrt{-g} = \sqrt{-g}\Gamma_{\rho}, \tag{2.47}
$$

therefore

$$\partial_\rho \boldsymbol{W}^{\kappa\rho,\alpha}{}_\gamma = \frac{\sqrt{-g}}{2\kappa}\Big[g^{\alpha\kappa}\Gamma_\gamma - \Gamma_\rho g^{\alpha\rho}\delta^\kappa_\gamma + \partial_\gamma g^{\alpha\kappa} - \partial_\rho g^{\alpha\rho}\delta^\kappa_\gamma\Big] \tag{2.48}$$

is obtained. Also,

$$\begin{aligned}\partial_\lambda g^{\alpha\beta} &= -g^{\mu\alpha}g^{\nu\beta}\partial_\lambda g_{\mu\nu} \\ &= -g^{\mu\alpha}g^{\nu\beta}(g_{\mu\rho}\Gamma^\rho{}_{\nu\lambda} + g_{\nu\rho}\Gamma^\rho{}_{\lambda\mu}) \\ &= -(g^{\nu\beta}\Gamma^\alpha{}_{\nu\lambda} + g^{\mu\alpha}\Gamma^\beta{}_{\lambda\mu}), & (2.49)\\ \partial_\lambda g^{\alpha\lambda} &= -(g^{\nu\lambda}\Gamma^\alpha{}_{\nu\lambda} + g^{\mu\alpha}\Gamma_\mu), & (2.50)\end{aligned}$$

it follows that

$$\begin{aligned}\partial_\rho \boldsymbol{W}^{\kappa\rho,\alpha}{}_\gamma &= \frac{\sqrt{-g}}{2\kappa}\Big[g^{\alpha\kappa}\Gamma_\gamma - \Gamma_\rho g^{\alpha\rho}\delta^\kappa_\gamma - g^{\nu\kappa}\Gamma^\alpha{}_{\nu\gamma} - g^{\mu\alpha}\Gamma^\kappa{}_{\gamma\mu} + (g^{\nu\lambda}\Gamma^\alpha{}_{\nu\lambda} + g^{\mu\alpha}\Gamma_\mu)\delta^\kappa_\gamma\Big] \\ &= \frac{\sqrt{-g}}{2\kappa}\Big[g^{\alpha\kappa}\Gamma_\gamma - g^{\nu\kappa}\Gamma^\alpha{}_{\nu\gamma} - g^{\mu\alpha}\Gamma^\kappa{}_{\gamma\mu} + g^{\nu\lambda}\Gamma^\alpha{}_{\nu\lambda}\delta^\kappa_\gamma\Big], & (2.51)\end{aligned}$$

therefore

$$\begin{aligned}\boldsymbol{f}^{\kappa\alpha}{}_\gamma &= \frac{1}{2\kappa}\sqrt{-g}\Big[\delta^\alpha_\gamma(g^{\mu\rho}\Gamma^\kappa{}_{\mu\rho} - g^{\kappa\lambda}\Gamma_\lambda) + \delta^\kappa_\gamma g^{\lambda\alpha}\Gamma_\lambda + g^{\kappa\alpha}\Gamma_\gamma - 2g^{\rho\alpha}\Gamma^\kappa{}_{\gamma\rho} \\ &\quad -g^{\alpha\kappa}\Gamma_\gamma + g^{\nu\kappa}\Gamma^\alpha{}_{\nu\gamma} + g^{\mu\alpha}\Gamma^\kappa{}_{\gamma\mu} - g^{\nu\lambda}\Gamma^\alpha{}_{\nu\lambda}\delta^\kappa_\gamma\Big] \\ &= \frac{1}{2\kappa}\sqrt{-g}\Big[\delta^\alpha_\gamma(g^{\mu\rho}\Gamma^\kappa{}_{\mu\rho} - g^{\kappa\lambda}\Gamma_\lambda) + \delta^\kappa_\gamma(g^{\lambda\alpha}\Gamma_\lambda - g^{\nu\lambda}\Gamma^\alpha{}_{\nu\lambda}) - g^{\rho\alpha}\Gamma^\kappa{}_{\gamma\rho} + g^{\nu\kappa}\Gamma^\alpha{}_{\nu\gamma}\Big]. & (2.52)\end{aligned}$$

The expression for $\boldsymbol{m}^{\kappa\rho}{}_\gamma$ is

$$\begin{aligned}\boldsymbol{m}^{\kappa\rho}{}_\gamma &= \frac{\sqrt{-g}}{2\kappa}\Big[\delta^\kappa_\gamma g^{\mu\nu}\Gamma^\alpha{}_{\mu\nu} + g^{\kappa\alpha}\Gamma_\gamma - 2g^{\rho\alpha}\Gamma^\kappa{}_{\gamma\rho} \\ &\quad -g^{\alpha\kappa}\Gamma_\gamma + g^{\nu\kappa}\Gamma^\alpha{}_{\nu\gamma} + g^{\mu\alpha}\Gamma^\kappa{}_{\gamma\mu} - g^{\nu\lambda}\Gamma^\alpha{}_{\nu\lambda}\delta^\kappa_\gamma\Big] \\ &= \frac{\sqrt{-g}}{2\kappa}\Big[g^{\nu\kappa}\Gamma^\alpha{}_{\nu\gamma} - g^{\rho\alpha}\Gamma^\kappa{}_{\gamma\rho}\Big]. & (2.53)\end{aligned}$$

For more advanced discussions of the energy of the gravitational field that could not be explained in this article, see [4].

2.3 Expression for Utiyama's Superpotential $\boldsymbol{U}^{\lambda\mu}{}_\nu$

We determine $\boldsymbol{U}^{\lambda\mu}{}_\nu$.

First, Equation (1.91) was

$$\boldsymbol{U}^{\lambda\mu}{}_\nu := -\frac{1}{\kappa}\delta^{[\lambda}_\nu \boldsymbol{D}^{\mu]} + \boldsymbol{C}^{[\lambda,\mu]}{}_\nu - \frac{1}{3}\partial_\rho(\boldsymbol{F}^{\lambda,\rho\mu}{}_\nu - \boldsymbol{F}^{\mu,\rho\lambda}{}_\nu). \tag{2.54}$$

From (2.40),

$$\boldsymbol{C}^{[\kappa,\alpha]}{}_{\gamma} = \frac{\sqrt{-g}}{2\kappa}\Big[\frac{1}{2}\delta^{\kappa}_{\gamma}g^{\mu\nu}\Gamma^{\alpha}{}_{\mu\nu} - \frac{1}{2}\delta^{\alpha}_{\gamma}g^{\mu\nu}\Gamma^{\kappa}{}_{\mu\nu} - g^{\rho\alpha}\Gamma^{\kappa}{}_{\gamma\rho} + g^{\rho\kappa}\Gamma^{\alpha}{}_{\gamma\rho}\Big] \tag{2.55}$$

holds. From (2.24),

$$\boldsymbol{F}^{\kappa,\rho\alpha}{}_{\gamma} = \frac{\sqrt{-g}}{4\kappa}\Big(- 2\delta^{\kappa}_{\gamma}g^{\rho\alpha} + \delta^{\rho}_{\gamma}g^{\kappa\alpha} + \delta^{\alpha}_{\gamma}g^{\kappa\rho}\Big), \tag{2.56}$$

therefore

$$\begin{aligned}\boldsymbol{F}^{\kappa,\rho\alpha}{}_{\gamma} - \boldsymbol{F}^{\alpha,\rho\kappa}{}_{\gamma} &= \frac{\sqrt{-g}}{4\kappa}\Big(- 2\delta^{\kappa}_{\gamma}g^{\rho\alpha} + 2\delta^{\alpha}_{\gamma}g^{\rho\kappa} + \delta^{\alpha}_{\gamma}g^{\kappa\rho} - \delta^{\kappa}_{\gamma}g^{\alpha\rho}\Big)\\ &= \frac{3}{4\kappa}\sqrt{-g}\Big(- \delta^{\kappa}_{\gamma}g^{\rho\alpha} + \delta^{\alpha}_{\gamma}g^{\rho\kappa}\Big)\end{aligned} \tag{2.57}$$

and

$$-\frac{1}{3}\partial_{\rho}(\boldsymbol{F}^{\kappa,\rho\alpha}{}_{\gamma} - \boldsymbol{F}^{\alpha,\rho\kappa}{}_{\gamma}) = \frac{1}{4\kappa}[\delta^{\kappa}_{\gamma}\partial_{\rho}(\sqrt{-g}g^{\rho\alpha}) - \delta^{\alpha}_{\gamma}\partial_{\rho}(\sqrt{-g}g^{\rho\kappa})]. \tag{2.58}$$

Here,

$$\partial_{\rho}(\sqrt{-g}g^{\rho\alpha}) = \sqrt{-g}(\Gamma_{\rho}g^{\rho\alpha} + \partial_{\rho}g^{\rho\alpha}) \tag{2.59}$$

and

$$\begin{aligned}\partial_{\rho}g^{\beta\alpha} &= -g^{\beta\mu}g^{\alpha\nu}\partial_{\rho}g_{\mu\nu}\\ &= -g^{\beta\mu}g^{\alpha\nu}(g_{\mu\delta}\Gamma^{\delta}{}_{\nu\rho} + g_{\nu\delta}\Gamma^{\delta}{}_{\rho\mu}),\end{aligned} \tag{2.60}$$

therefore

$$\begin{aligned}\partial_{\rho}(\sqrt{-g}g^{\rho\alpha}) &= \sqrt{-g}(\Gamma_{\rho}g^{\rho\alpha} - g^{\rho\mu}g^{\alpha\nu}g_{\mu\delta}\Gamma^{\delta}{}_{\nu\rho} - g^{\rho\mu}g^{\alpha\nu}g_{\nu\delta}\Gamma^{\delta}{}_{\rho\mu})\\ &= \sqrt{-g}(\Gamma_{\rho}g^{\rho\alpha} - g^{\alpha\nu}\Gamma^{\rho}{}_{\nu\rho} - g^{\rho\mu}\Gamma^{\alpha}{}_{\rho\mu})\\ &= -\sqrt{-g}g^{\rho\mu}\Gamma^{\alpha}{}_{\rho\mu}\end{aligned} \tag{2.61}$$

holds. Therefore,

$$-\frac{1}{3}\partial_{\rho}(\boldsymbol{F}^{\kappa,\rho\alpha}{}_{\gamma} - \boldsymbol{F}^{\alpha,\rho\kappa}{}_{\gamma}) = \frac{\sqrt{-g}}{4\kappa}[-\delta^{\kappa}_{\gamma}g^{\mu\nu}\Gamma^{\alpha}{}_{\mu\nu} + \delta^{\alpha}_{\gamma}g^{\mu\nu}\Gamma^{\kappa}{}_{\mu\nu}] \tag{2.62}$$

holds. From this,

$$\begin{aligned}&\boldsymbol{C}^{[\kappa,\alpha]}{}_{\gamma} - \frac{1}{3}\partial_{\rho}(\boldsymbol{F}^{\kappa,\rho\alpha}{}_{\gamma} - \boldsymbol{F}^{\alpha,\rho\kappa}{}_{\gamma})\\ &= \frac{\sqrt{-g}}{4\kappa}\Big[\delta^{\kappa}_{\gamma}g^{\mu\nu}\Gamma^{\alpha}{}_{\mu\nu} - \delta^{\alpha}_{\gamma}g^{\mu\nu}\Gamma^{\kappa}{}_{\mu\nu} - 2g^{\rho\alpha}\Gamma^{\kappa}{}_{\gamma\rho} + 2g^{\rho\kappa}\Gamma^{\alpha}{}_{\gamma\rho}\\ &\qquad -\delta^{\kappa}_{\gamma}g^{\mu\nu}\Gamma^{\alpha}{}_{\mu\nu} + \delta^{\alpha}_{\gamma}g^{\mu\nu}\Gamma^{\kappa}{}_{\mu\nu}\Big]\\ &= \frac{\sqrt{-g}}{2\kappa}\Big[- g^{\rho\alpha}\Gamma^{\kappa}{}_{\gamma\rho} + g^{\rho\kappa}\Gamma^{\alpha}{}_{\gamma\rho}\Big]\end{aligned} \tag{2.63}$$

is obtained. Therefore,

$$\boldsymbol{U}^{\kappa\alpha}{}_{\gamma} = \frac{\sqrt{-g}}{2\kappa}\Big[- g^{\rho\alpha}\Gamma^{\kappa}{}_{\gamma\rho} + g^{\rho\kappa}\Gamma^{\alpha}{}_{\gamma\rho} - \delta^{\kappa}_{\gamma}(g^{\mu\nu}\Gamma^{\alpha}{}_{\mu\nu} - g^{\mu\alpha}\Gamma_{\mu}) + \delta^{\alpha}_{\gamma}(g^{\mu\nu}\Gamma^{\kappa}{}_{\mu\nu} - g^{\mu\kappa}\Gamma_{\mu})\Big] \tag{2.64}$$

This agrees with the Freud superpotential $\boldsymbol{f}^{\kappa\alpha}_{\gamma}$.

A Einstein Energy Pseudotensor $\boldsymbol{t}^{\mu}{}_{\nu}$

We determine $\boldsymbol{t}^{\mu}{}_{\nu}$:

$$\boldsymbol{t}^{\mu}{}_{\nu} = \frac{1}{2\kappa}\Big(\frac{\partial \boldsymbol{G}}{\partial(\partial_\mu g_{\alpha\beta})}\partial_\nu g_{\alpha\beta} - \delta^\mu_\nu \boldsymbol{G}\Big). \tag{A.1}$$

Here, we put

$$t^{\mu}{}_{\nu} := \frac{\boldsymbol{t}^{\mu}{}_{\nu}}{\sqrt{-g}} =: \frac{1}{2\kappa}(s^{\mu}{}_{\nu} - \delta^\mu_\nu G), \tag{A.2}$$

then

$$\begin{aligned} s^{\mu}{}_{\nu} &= \frac{\partial G}{\partial(\partial_\mu g_{\alpha\beta})}\partial_\nu g_{\alpha\beta} \\ &= G^{\mu,\alpha\beta}\partial_\nu g_{\alpha\beta}. \end{aligned} \tag{A.3}$$

From (2.19) and (2.35), we obtain

$$s^{\sigma}{}_{\nu} = \frac{1}{2}\Big[\Gamma^{\sigma}{}_{\mu\rho}(2g^{\mu\tau}g^{\rho\delta} - g^{\mu\rho}g^{\delta\tau}) + (g^{\delta\tau}g^{\sigma\gamma} - g^{\sigma\tau}g^{\gamma\delta} - g^{\sigma\delta}g^{\gamma\tau})\Gamma_\gamma\Big](g_{\delta\alpha}\Gamma^{\alpha}{}_{\tau\nu} + g_{\tau\alpha}\Gamma^{\alpha}{}_{\nu\delta}). \tag{A.4}$$

Hence,

$$s^{\sigma\beta} := g^{\beta\nu}s^{\sigma}{}_{\nu} = {}^{(1)}s^{\sigma\beta} + {}^{(2)}s^{\sigma\beta}, \tag{A.5}$$

$${}^{(1)}s^{\sigma\beta} = \frac{1}{2}\Gamma^{\sigma}{}_{\mu\rho}(2g^{\mu\tau}g^{\rho\delta} - g^{\mu\rho}g^{\delta\tau})g^{\beta\nu}(g_{\delta\alpha}\Gamma^{\alpha}{}_{\tau\nu} + g_{\tau\alpha}\Gamma^{\alpha}{}_{\nu\delta}), \tag{A.6}$$

$${}^{(2)}s^{\sigma\beta} = \frac{1}{2}(g^{\delta\tau}g^{\sigma\gamma} - g^{\sigma\tau}g^{\gamma\delta} - g^{\sigma\delta}g^{\gamma\tau})\Gamma_\gamma g^{\beta\nu}(g_{\delta\alpha}\Gamma^{\alpha}{}_{\tau\nu} + g_{\tau\alpha}\Gamma^{\alpha}{}_{\nu\delta}). \tag{A.7}$$

First,

$$\begin{aligned} 2\,{}^{(1)}s^{\sigma\beta} &= 2\Gamma^{\sigma}{}_{\mu\rho}\Gamma^{\rho}{}_{\tau\nu}g^{\mu\tau}g^{\beta\nu} + 2\Gamma^{\sigma}{}_{\mu\rho}\Gamma^{\mu}{}_{\nu\delta}g^{\beta\nu}g^{\rho\delta} \\ &\quad -\Gamma^{\sigma}{}_{\mu\rho}\Gamma_\nu g^{\mu\rho}g^{\beta\nu} - \Gamma^{\sigma}{}_{\mu\rho}\Gamma_\nu g^{\mu\rho}g^{\beta\nu}, \end{aligned} \tag{A.8}$$

$${}^{(1)}s^{\sigma\beta} = 2\Gamma^{\sigma}{}_{\mu\rho}\Gamma^{\rho}{}_{\tau\nu}g^{\mu\tau}g^{\beta\nu} - \Gamma^{\sigma}{}_{\mu\rho}\Gamma_\nu g^{\mu\rho}g^{\beta\nu}. \tag{A.9}$$

Second,

$$\begin{aligned} 2\,{}^{(2)}s^{\sigma\beta} &= \Gamma_\gamma\Gamma_\nu g^{\sigma\gamma}g^{\beta\nu} + \Gamma_\gamma\Gamma_\nu g^{\sigma\gamma}g^{\beta\nu} \\ &\quad -\Gamma_\gamma\Gamma^{\gamma}{}_{\tau\nu}g^{\sigma\tau}g^{\beta\nu} - \Gamma_\gamma\Gamma^{\sigma}{}_{\nu\delta}g^{\beta\nu}g^{\gamma\delta} \\ &\quad -\Gamma_\gamma\Gamma^{\sigma}{}_{\tau\nu}g^{\gamma\tau}g^{\beta\nu} - \Gamma_\gamma\Gamma^{\gamma}{}_{\nu\delta}g^{\sigma\delta}g^{\beta\nu}, \end{aligned} \tag{A.10}$$

$${}^{(2)}s^{\sigma\beta} = \Gamma_\gamma\Gamma_\nu g^{\sigma\gamma}g^{\beta\nu} - \Gamma_\gamma\Gamma^{\gamma}{}_{\tau\nu}g^{\sigma\tau}g^{\beta\nu} - \Gamma_\gamma\Gamma^{\sigma}{}_{\nu\delta}g^{\beta\nu}g^{\gamma\delta}. \tag{A.11}$$

Then, we obtain

$$s^{\sigma\beta} = 2\Gamma^{\sigma}{}_{\mu\rho}\Gamma^{\rho}{}_{\tau\nu}g^{\mu\tau}g^{\beta\nu} - \Gamma^{\sigma}{}_{\mu\rho}\Gamma_\nu g^{\mu\rho}g^{\beta\nu} - \Gamma_\gamma\Gamma^{\sigma}{}_{\nu\delta}g^{\beta\nu}g^{\gamma\delta} + \Gamma_\gamma\Gamma_\nu g^{\sigma\gamma}g^{\beta\nu} - \Gamma_\gamma\Gamma^{\gamma}{}_{\tau\nu}g^{\sigma\tau}g^{\beta\nu}. \tag{A.12}$$

Note note $s^{\mu\nu} \neq s^{\nu\mu}$.

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